

# A CHIRAL APERIODIC POLYGON WITH FIBONACCI MONODROMY

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**ABSTRACT.** We introduce an explicit 27-gon and prove that it is a strictly chiral aperiodic monotile. It is  $\text{Tile}(1, 1)$  with two-segment profiles on its edges; every tiling by congruent copies, reflections included, is homochiral and non-periodic. We show that, on an invariant theta graph, the boundary substitution of its supertiles is the cube of the Fibonacci automorphism of the rank-two free group up to inner automorphism and inversion of generators, so its mapping torus is homotopy equivalent to a non-orientable once-punctured torus bundle whose even-degree covers are cyclic covers of the figure-eight knot complement.

We prove that these boundaries carry the arithmetic of the golden ratio  $\phi$ . The cover of degree  $n$  has torsion homology  $\mathbb{Z}[\phi]/((-\phi)^{3n} - 1)$ , of order the Lucas number  $L_{3n}$  or  $L_{3n} - 2$  for odd or even  $n$ ; the roots  $\phi$  and  $-1/\phi$  alternate with the orientation, and the quotients  $\mathbb{Z}[\phi]/(\phi^p - 1)$  at primes  $p \geq 5$  are fields exactly when  $L_p$  is prime. The rescaled boundaries of the even-level singleton supertiles converge to a Jordan curve of Hausdorff dimension  $6 \log \phi / \log(4 + \sqrt{15})$ . The tile thus retains both directions of the Fibonacci recurrence in one handedness, and its boundary dimension is the growth rate of the Lucas numbers against its expansion.

## 1. INTRODUCTION

The polygon  $\text{Tile}(1, 1)$ , the equilateral member of the family  $\text{Tile}(a, b)$  introduced with the hat [20], has fourteen unit edges, two of them collinear, and is a weakly chiral aperiodic monotile: its homochiral tilings are non-periodic, while tilings that mix the two handednesses can be periodic [21]. The same authors obtained the Spectres by replacing every edge of  $\text{Tile}(1, 1)$  with a copy of one simple, non-straight, smooth curve, alternately facing inwards and outwards, and proved that the resulting shapes are strictly chiral aperiodic monotiles: every tiling by a Spectre, reflected copies permitted, is homochiral and non-periodic. Their proof uses the smoothness of the curve where two tiles meet and applies to  $C^1$  paths; the closing section of [21] leaves open whether piecewise-linear paths might permit different tilings. Theorem A answers this for the two-segment paths of the parameter set  $U$  below: they do not, and the resulting 27-gons are Spectres.

For  $c = x - it$  put

$$U = \left\{ x - it : \left| x - \frac{3}{4} \right| \leq \frac{1}{100}, \left| t - \frac{1}{4} \right| \leq \frac{1}{100} \right\}, \quad c_0 = \frac{3 - i}{4}.$$

Let  $P_c$  be the polygon obtained from  $\text{Tile}(1, 1)$  by replacing each even-numbered edge with the two-segment profile  $(0, c, 1)$  and each odd-numbered edge with the complementary profile  $(0, 1 - c, 1)$ , in the coordinates fixed in Section 2; the fourteen two-segment paths are the *profiled edges* of  $P_c$ . A *tiling* is a locally finite covering of the plane by congruent copies of a compact tile with pairwise disjoint interiors; all Euclidean isometries, reflections included, are allowed. A tiling is *homochiral* if all its tiles have one handedness and *non-periodic* if it has no nonzero translation symmetry. A tile is a *strictly chiral aperiodic monotile* if it admits a tiling and every tiling by it is homochiral and non-periodic [21].

**Theorem A** (The polygon). *For every  $c \in U$  the region  $P_c$  is a simple 27-gon with trivial symmetry group, and it is a strictly chiral aperiodic monotile. Every tiling by congruent copies of  $P_c$  is edge-to-edge with respect to the profiled edges: each profiled edge of a tile coincides with a complementary profiled edge of a neighbouring tile of the same handedness. Straightening the profiles is a mutual local derivability between the tilings of  $P_c$  and the homochiral tilings of  $\text{Tile}(1, 1)$ , and for  $c, d \in U$  the tilings of  $P_c$  and  $P_d$  are mutually locally derivable.*

The homochiral tilings of  $\text{Tile}(1, 1)$  carry the hierarchy of [21]: every tile belongs to a unique nested sequence of supertiles, recognized from bounded neighbourhoods. By Theorem A the tilings of  $P_c$  carry the same hierarchy. The linear expansion over two consecutive levels is

$$\lambda = 4 + \sqrt{15},$$

the fundamental unit of  $\mathbb{Q}(\sqrt{15})$ , and the number of exposed profiled edges on the boundary of a supertile grows by the factor

$$\theta = \phi^3 = 2 + \sqrt{5}, \quad \phi = \frac{1 + \sqrt{5}}{2},$$

at each level, with  $\phi$  the fundamental unit of  $\mathbb{Q}(\sqrt{5})$ . The boundary of a supertile of level  $m$ , with the four arcs between its key points named  $A, B, C, D$  in cyclic order, is subdivided by the supertile's children according to the signed rule

$$A \mapsto \overline{B} \overline{C} \overline{B}, \quad B \mapsto \overline{A} \overline{B} \overline{C}, \quad C \mapsto \overline{B} \overline{C} \overline{B} \overline{A} \overline{B} \overline{C} \overline{B}, \quad D \mapsto \overline{A} \overline{D} \overline{C}, \quad (1.1)$$

in which a bar denotes reversal of traversal and the Euclidean positions of the child arcs are given in Section 4.

**Theorem B** (The boundary of the supertiles). *Every level of the hierarchy subdivides the boundary of a singleton supertile by (1.1). Rescaled by the expansion and read in the frame of its key points, the boundary of every singleton supertile of even level converges in Hausdorff distance to a single Jordan curve  $K$ , at rate  $O(\lambda^{-m})$  at level  $2m$ , uniformly in  $c \in U$ , the tiling and the supertile; the Hausdorff and box dimensions of  $K$  are*

$$\delta = \frac{\log \theta^2}{\log \lambda} = \frac{6 \log \phi}{\log(4 + \sqrt{15})}, \quad 0 < \mathcal{H}^\delta(K) < \infty.$$

The dimension is six times the ratio of the regulators of  $\mathbb{Q}(\sqrt{5})$  and  $\mathbb{Q}(\sqrt{15})$ : the boundary grows in one real quadratic field and the tile expands in another.

The rule (1.1) records the order and the direction of the child arcs, not only their number: identify the initial and terminal key points of alternate arcs, so that  $A$  and  $C$  run from a vertex  $v$  to a vertex  $w$  and  $B$  and  $D$  run back. The edges  $A, B, C$  span an invariant theta graph  $\Theta$  with fundamental group free on

$$a = AB, \quad c = CB.$$

Let  $q$  be the Fibonacci automorphism  $q(a) = c$ ,  $q(c) = ac$  of the free group  $F(a, c)$ , let  $\iota$  be the automorphism inverting both generators, and write  $\text{Inn}(g)$  for conjugation  $x \mapsto gxg^{-1}$ .

**Theorem C** (Fibonacci monodromy). *Read on  $\Theta$ , one level of the subdivision (1.1) induces the automorphism*

$$\rho = \text{Inn}(u^{-1}) \circ \iota \circ q^3, \quad u = cac,$$

*of  $F(a, c)$ . Its image in  $\text{Out } F(a, c) = \text{GL}_2(\mathbb{Z})$  is  $-Q^3$  with  $Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ , and  $\rho^2 = \text{Inn}(u q^3(u)) \circ q^6$ .*

*The retraction of the full endpoint graph onto  $\Theta$  sending  $D$  to  $B$  commutes with the subdivision.*

Let  $T$  be the mapping torus of the subdivision on  $\Theta$  and  $T_n$  its  $n$ -fold cyclic cover in the suspension direction; the  $T_n$  with their covering maps form the *tower of cyclic covers* of  $T$ . For  $A \in \text{GL}_2(\mathbb{Z})$  let  $M(A)$  denote the once-punctured torus bundle with monodromy  $A$ ; it is orientable exactly when  $\det A = 1$ . The bundle  $M(Q)$  is the Gieseking manifold and  $M(Q^2)$  is the complement of the figure-eight knot.

**Theorem D** (The tower of cyclic covers). *There are homotopy equivalences  $T_n \simeq M((-Q)^{3n})$ , compatible with all covering maps between the  $T_n$ . The bundle is homeomorphic to the  $3n$ -fold cyclic cover of the Gieseking manifold; it is orientable exactly when  $n$  is even, and for  $n = 2m$  it*

is the  $3m$ -fold cyclic cover of the figure-eight knot complement. Write  $\mathcal{O} = \mathbb{Z}[\phi]$  and  $\psi = -1/\phi$  for the conjugate root of  $x^2 = x + 1$ , and let  $\mathcal{O}$  act on the cycle lattice of  $\Theta$  so that the Fibonacci automorphism acts as multiplication by its expanding eigenvalue  $\phi$ . Then

$$\mathrm{Tor} H_1(T_n; \mathbb{Z}) \cong \mathcal{O} / ((-\phi)^{3n} - 1) = \begin{cases} \mathcal{O} / (\phi^{3n} - 1), & n \text{ even}, \\ \mathcal{O} / (\psi^{3n} - 1), & n \text{ odd}, \end{cases}$$

as  $\mathcal{O}$ -modules. For  $m \mid n$  the covering projection  $T_n \rightarrow T_m$  acts on these modules by reduction and the transfer acts by multiplication by  $\sum_{j < n/m} (-\phi)^{3mj}$ .

With the cycle lattice marked by  $\phi$ , the annihilator of the torsion is  $(\phi^{3n} - 1)$  at the orientable levels and  $(\psi^{3n} - 1)$  at the non-orientable ones: the two roots of  $x^2 = x + 1$  alternate with the orientation, and the orientation character of the bundle is the norm of the monodromy unit  $(-\phi)^{3n}$ . Write  $\mathcal{O}_j = \mathcal{O} / (\phi^j - 1)$ , and let  $F_j$  and  $L_j$  be the Fibonacci and Lucas numbers.

**Theorem E** (The arithmetic of the tower). (i)  $\mathcal{O}_j$  is the cokernel of  $Q^j - I$  on  $\mathbb{Z}^2$ , the group of fixed points of  $Q^j$  on  $\mathbb{R}^2/\mathbb{Z}^2$ , and the abelianization of the Fibonacci group  $F(2, j)$ ; for even  $j$  it is the torsion homology of the  $(j/2)$ -fold cyclic cover of the figure-eight knot complement. (ii)  $|\mathcal{O}_j| = |L_j - 1 - (-1)^j|$ . The torsion homology of  $T_n$  has order  $L_{3n}$  for odd  $n$  and  $L_{3n} - 2$  for even  $n$ , with invariant factors  $(2, L_{3n}/2)$  for odd  $n$ ,  $(L_{3m}, L_{3m})$  for  $n = 2m$  with  $m$  odd, and  $(F_{3m}, 5F_{3m})$  for  $n = 2m$  with  $m$  even. (iii) For a prime  $p \neq 2, 5$ , the levels  $n$  at which  $T_n$  has  $p$ -torsion are the multiples of the least order of  $-\phi^3$  modulo the prime ideals of  $\mathcal{O}$  above  $p$ ; on the orientable levels  $n = 2m$  they are those with  $m$  a multiple of the order of  $\phi^6$  in  $(\mathcal{O}/p\mathcal{O})^\times$ , which divides  $p - (\frac{5}{p})$ . (iv)  $\mathcal{O}_j$  is a field exactly when  $j \in \{3, 4\}$  or  $j = p \geq 5$  is a prime with  $L_p$  prime. The tower reaches no prime index  $p \geq 5$ ; exactly one of its torsion rings is a field,  $\mathbb{F}_4 = \mathrm{Tor} H_1(T_1; \mathbb{Z})$  at index 3, and every torsion ring reduces modulo 2 to  $\mathbb{F}_4$ . (v) The boundary dimension is the growth rate of the torsion against the expansion of one level:

$$\delta = \lim_{n \rightarrow \infty} \frac{\log |\mathrm{Tor} H_1(T_n; \mathbb{Z})|}{n \log \sqrt{\lambda}} = \lim_{n \rightarrow \infty} \frac{\log L_{3n}}{n \log \sqrt{\lambda}} = \frac{3 \log M(t^2 - 3t + 1)}{\log \lambda},$$

with  $M$  the Mahler measure and  $t^2 - 3t + 1$  the Alexander polynomial of the figure-eight knot.

Sections 2–4 prove Theorems A and B, Sections 5–6 Theorems C and D, and Section 7 Theorem E; the appendices record the exact finite computations on which the geometric proofs rest. To our knowledge, Theorems C–E are the first identification of the theta-graph suspension of a monotile's boundary substitution, up to homotopy, with a cyclic cover of the Gieseking manifold, and of the torsion homology of its cyclic covers with the abelianizations of the Fibonacci groups; they concern the boundary of the hierarchy rather than the tiling space, whose Čech cohomology is computed in [2].

The profiles of  $P_c$  are the piecewise-linear case of the alternating construction of [21], an asymmetric two-segment path on the even edges paired with its complement on the odd edges, and the proof of Theorem A uses the strict inequality  $\ell > s$  between the two legs, which forces a reflected contact to end a short leg inside a long one (Lemma 2.3). Equal-leg profiles have  $\ell = s$  and are not covered by this proof; Goucher's equilateral polygon, with two segments of equal length on each edge of  $\mathrm{Tile}(1, 1)$  and vertices in  $\mathbb{Z}[e^{\pi i/6}]$ , is one such polygon,<sup>1</sup> and no proof has been published for it or for any other equal-leg profile. To our knowledge,  $P_c$  is the first polygon proved in print to be a strictly chiral aperiodic monotile, and Corollary 3.4 makes it the first polygonal Spectre in the same sense.

<sup>1</sup><https://cp4space.hatsya.com/2023/07/23/miscellaneous-discoveries/>, accessed 28 September 2026; its comments link an argument for its chirality, <https://twitter.com/apgox/status/1663525666594201601>.

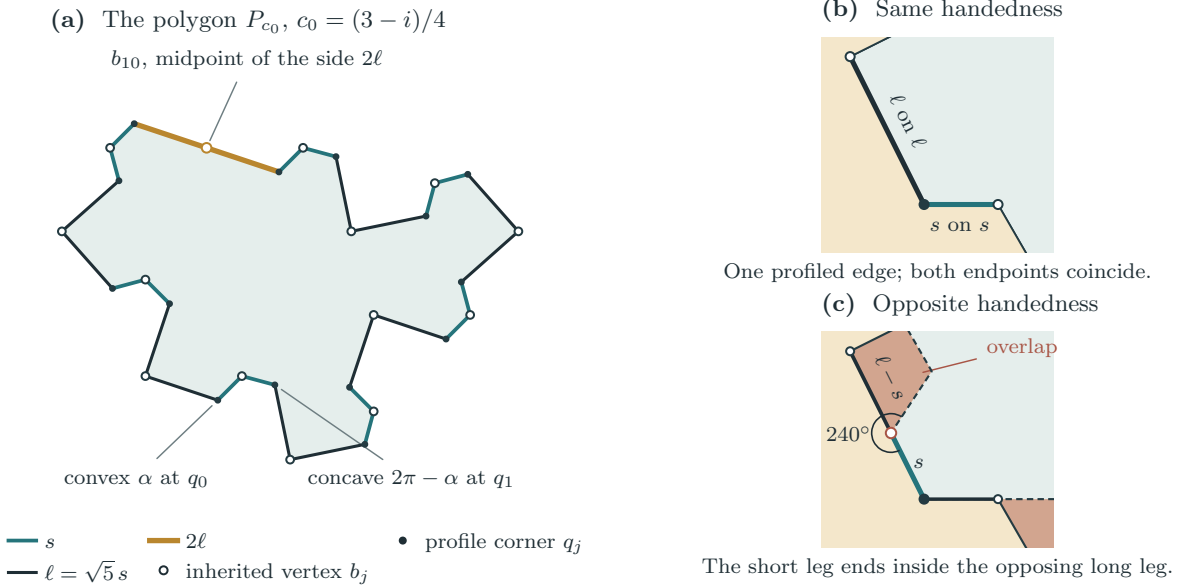


FIGURE 1. The polygon  $P_{c_0}$ . (a) Inherited vertices  $b_j$ , profile corners  $q_j$ , the mark  $b_{10}$ , the legs of length  $s$  and  $\ell$ , and the maximal side of length  $2\ell$ . (b) A profiled edge shared by two tiles of the same handedness, as in the supertile of Figure 2(a). (c) A reflected contact: the short leg ends inside the long leg at a vertex of angle  $240^\circ$ , and the tiles overlap.

## 2. A STRICTLY CHIRAL APERIODIC POLYGON

2.1. **Coordinates.** Put  $\omega = e^{\pi i/6}$  and define the closed polygonal path

$$b_0 = 0, \quad b_{j+1} = b_j + \omega^{m_j}, \quad (m_j)_{j=0}^{13} = (0, 10, 1, 3, 0, 2, 5, 7, 4, 6, 6, 8, 11, 9), \quad (2.1)$$

so that  $b_{14} = b_0$ . Let  $H$  be the closed region it bounds. The point  $b_{10}$  lies in the interior of a straight side and is retained as a mark; thus  $H$  has fourteen marked unit edges and thirteen maximal sides. In these coordinates  $H$  is the polygon  $\text{Tile}(1, 1)$  of [21]. The interior angles at  $b_0, \dots, b_{13}$  are

$$(90, 240, 90, 120, 270, 120, 90, 120, 270, 120, 180, 120, 90, 240)^\circ. \quad (2.2)$$

For a path  $\gamma$  from 0 to 1 the *complementary path* is  $\gamma^*(\tau) = 1 - \gamma(1 - \tau)$ . We use the two-segment path  $\gamma_c = (0, c, 1)$  with

$$c = x - it \in U = \left\{ \left| x - \frac{3}{4} \right| \leq \frac{1}{100}, \left| t - \frac{1}{4} \right| \leq \frac{1}{100} \right\}, \quad c_0 = \frac{3 - i}{4}. \quad (2.3)$$

Replace edge  $j$  of  $H$  by  $b_j + \omega^{m_j} \gamma_c$  when  $j$  is even and by  $b_j + \omega^{m_j} \gamma_c^*$  when  $j$  is odd. With

$$q_j = \begin{cases} b_j + \omega^{m_j} c, & j \text{ even,} \\ b_j + \omega^{m_j} (1 - c), & j \text{ odd,} \end{cases} \quad (2.4)$$

the new boundary is the closed path  $(b_0, q_0, b_1, q_1, \dots, b_{13}, q_{13}, b_0)$ , and  $P_c$  is the region it bounds. The two-segment path from  $b_j$  to  $b_{j+1}$  is the *profiled edge*  $j$ , its middle vertex  $q_j$  is a *profile corner*, the  $b_j$  are the *inherited vertices*, and the two segments of a profiled edge are its *legs*, of lengths

$$\ell = |c|, \quad s = |1 - c|.$$

On an even profiled edge the long leg comes first in the positive traversal; on an odd one the short leg comes first.

## 2.2. The geometry of $P_c$ .

**Proposition 2.1.** *For every  $c \in U$  the region  $P_c$  is a simple 27-gon with trivial symmetry group. Its maximal sides have lengths  $s$ ,  $\ell$  and  $2\ell$  with multiplicities 14, 12 and 1, and*

$$|P_c| = 3 + 3\sqrt{3}, \quad |\partial P_c| = 14(\ell + s). \quad (2.5)$$

*The interior angle at an even profile corner is*

$$\alpha = \pi - \arg \frac{1-c}{c} \in \left( \frac{\pi}{2}, \frac{2\pi}{3} \right), \quad (2.6)$$

*that at an odd profile corner is  $2\pi - \alpha$ , and the angles at  $b_0, \dots, b_{13}$  are those of (2.2). At  $c = c_0$ ,  $s = \sqrt{2}/4$  and  $\ell = \sqrt{5}s$ .*

*Proof. Angles and sides.* Write  $c = x - it$  and  $D = x(1-x) - t^2$ . On  $U$ ,

$$\ell^2 - s^2 = 2x - 1 > 0, \quad 0 < \frac{287}{2500} \leq D \leq \frac{337}{2500} < \frac{t}{\sqrt{3}}, \quad (2.7)$$

the extreme values of  $D$  being attained at the corners of  $U$  because  $D$  decreases in  $x$  and in  $t$  there, and  $3(337/2500)^2 < (24/100)^2$ . Hence  $\ell > s$ , and  $\arg((1-c)/c) \in (\pi/3, \pi/2)$  since  $(1-c)/c = (D+it)/|c|^2$ , which is (2.6). At  $b_j$  the two adjacent legs are the two adjacent edges of  $H$  rotated through one angle, so the angle is that of  $H$ . The only angle of  $180^\circ$  is at  $b_{10}$ , where two long legs form the single side of length  $2\ell$ ; the remaining sides are single legs.

*Simplicity.* At  $c = c_0$  all vertices have coordinates in  $\mathbb{Q}(\sqrt{3})$ , and by Proposition A.1 every pair of nonadjacent maximal sides is separated, in the  $x$ - or in the  $y$ -coordinate, by a gap of at least  $(3 - \sqrt{3})/8 > 3/20$ . Each  $q_j$  depends on  $c$  through an isometry or its negative, so

$$|q_j(c) - q_j(c_0)| \leq |c - c_0| \leq \frac{\sqrt{2}}{100} \quad (c \in U), \quad (2.8)$$

and a coordinate gap decreases by at most  $2\sqrt{2}/100 < 3/100$ . Nonadjacent sides therefore stay separated on  $U$ , and adjacent sides meet only at their common vertex since the turn there is nonzero. Thus the boundary path is simple.

*Area and perimeter.* The profile triangles on the even and on the odd edges have signed areas  $\pm t/2$  of opposite signs, so their contributions to the shoelace formula cancel and  $|P_c| = |H| = 3 + 3\sqrt{3}$ ; positivity shows that the displayed traversal is positive. The perimeter is fourteen profiled edges of length  $\ell + s$ .

*Symmetry.* A symmetry preserves the unique side of length  $2\ell$ . A nontrivial rotation preserving a segment is the half-turn about its midpoint, which exchanges the two sides of the segment and so cannot preserve  $P_c$ . A reflection sends convex profile corners to convex profile corners,  $\alpha$  being the only angle in  $(90^\circ, 120^\circ)$ ; in the positive traversal every convex corner is entered along a side of length  $\ell$  or  $2\ell$  and left along a side of length  $s$ , and a reflection reverses this order, which is impossible since  $\ell \neq s$ .  $\square$

**2.3. Edge-to-edge matching is forced.** Fix a tiling by congruent copies of  $P_c$ . By local finiteness the tiles meeting a point  $z$  divide a small disc about  $z$  into sectors with disjoint interiors whose angles sum to  $2\pi$ ; let  $\Sigma(z)$  be the multiset of these angles. Put

$$\Omega = \left\{ \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2} \right\},$$

the angles available at inherited vertices and at interior points of sides.

**Lemma 2.2** (Sectors). *For every  $z$  on the boundary of a tile, either  $\Sigma(z) \subset \Omega$  or  $\Sigma(z) = \{\alpha, 2\pi - \alpha\}$ . If  $z$  lies in the relative interior of a side of a tile and is a non-flat vertex of another tile, then  $\Sigma(z) = \{\pi, \pi/2, \pi/2\}$ .*

*Proof.* Let  $k$  and  $k'$  be the numbers of sectors of angle  $\alpha$  and  $2\pi - \alpha$  in  $\Sigma(z)$ , and let  $\beta$  be the sum of the remaining, plain, angles; then  $k\alpha + k'(2\pi - \alpha) + \beta = 2\pi$ , and  $\beta$  is a sum of elements of  $\Omega$ , so  $\beta = 0$  or  $\beta \geq \pi/2$ , and  $\beta \geq \pi$  if it has at least two summands. Since  $2\pi - \alpha > \pi$ ,  $k' \leq 1$ . If  $k' = 1$  then  $k\alpha + \beta = \alpha$ , so  $k \leq 1$ ;  $k = 1$  gives  $\beta = 0$ , and  $k = 0$  gives  $\beta = \alpha \in (\pi/2, 2\pi/3)$ , which is neither in  $\Omega$  nor a sum of two or more of its elements. If  $k' = 0$  then  $\beta = 2\pi - k\alpha$  with  $k \leq 3$ , since  $4\alpha > 2\pi$ . For  $k = 3$ ,  $\beta \in (0, \pi/2)$ ; for  $k = 2$ ,  $\beta \in (2\pi/3, \pi)$ ; for  $k = 1$ ,  $\beta \in (4\pi/3, 3\pi/2)$ . None of these open intervals contains an element of  $\Omega$ ; sums of two elements of  $\Omega$  lie in  $\{\pi, 7\pi/6, 4\pi/3, 3\pi/2, 5\pi/3, \dots\}$  and avoid them as well; and sums of three or more are at least  $3\pi/2$ . Hence  $k = k' = 0$ , or  $k = k' = 1$  and  $\beta = 0$ .

For the second statement,  $\pi \in \Sigma(z)$  excludes the case  $\{\alpha, 2\pi - \alpha\}$ , so the angle of the second tile at  $z$  lies in  $\Omega \cap (0, \pi) = \{\pi/2, 2\pi/3\}$ , and the remaining angle  $\pi/2$  or  $\pi/3$  is a sum of elements of  $\Omega$  only in the first case.  $\square$

**Lemma 2.3** (Forced edge-to-edge matching). *Every profiled edge of a tile coincides with a profiled edge of a neighbouring tile of the same handedness, traversed in the opposite direction. Consequently the tiling is homochiral.*

*Proof.* By Lemma 2.2 a convex profile corner of a tile  $X$  meets exactly one concave profile corner, of a tile  $Y$ , and no other sector; the two rays of the one corner are the two rays of the other. If  $X$  and  $Y$  have the same handedness, the long leg of  $X$  lies along the long leg of  $Y$  and the short leg along the short leg, so the profiled edges coincide. If they have opposite handedness, the short leg of one lies along the long leg of the other, and since  $s < \ell$  the inherited endpoint of the short leg lies in the relative interior of a side of the other tile. A short leg ends at  $b_{j+1}$  for even  $j$  and begins at  $b_j$  for odd  $j$ , so this endpoint has angle  $120^\circ$  or  $240^\circ$  by (2.2), contradicting the second statement of Lemma 2.2.

The profiled edges exhaust  $\partial P_c$ , the side of length  $2\ell$  being two legs, so the handedness agrees across every contact of positive length; and the graph whose vertices are the tiles and whose edges join tiles sharing such a contact is connected, since a path between two tile interiors can avoid the locally finite set of vertices. Hence all tiles have one handedness.  $\square$

**2.4. Straightening.** The homochiral tilings by  $H$  have the following alignment property, proved in [21, Lemma 2.1]: the fourteen marked unit edges, the mark  $b_{10}$  included, are paired as whole edges, and at a common vertex the inherited parities agree, so an even-numbered edge always meets an odd-numbered edge with opposite traversal.

**Lemma 2.4** (Winding-number criterion). *Let  $(X_i)$  be a locally finite family of compact simple polygons with positively oriented boundaries. If  $\sum_i \text{wind}(\partial X_i, z) = 1$  for every  $z$  outside  $\bigcup_i \partial X_i$ , then  $(X_i)$  is a tiling of the plane.*

*Proof.*  $\text{wind}(\partial X_i, z)$  is the indicator of the interior of  $X_i$ , so every  $z$  off the boundaries lies in exactly one interior. Two interiors cannot overlap, an overlap being open and hence containing such a  $z$ . The union  $\bigcup X_i$  is closed by local finiteness; its complement is open and contains no point off the boundaries, every such point lying in some  $X_i$ , so it is contained in the nowhere dense set  $\bigcup \partial X_i$  and is empty.  $\square$

A map between two tiling spaces is *local* if there is a radius  $r$  such that the tiles of the image tiling within distance 1 of any point are determined by the tiles of the source tiling within distance  $r$  of that point; two tiling spaces are *mutually locally derivable* (MLD) if there are inverse local maps between them commuting with the Euclidean isometries [3].

Put  $D_j(c) = \text{conv}\{b_j, q_j, b_{j+1}\}$  and  $D(c) = \bigcup_j D_j(c)$ . Then

$$P_c \triangle H \subset D(c), \quad d_H(P_c, H) \leq t \leq \frac{13}{50} : \quad (2.9)$$

the oriented boundaries of  $P_c$  and  $H$  differ by the fourteen triangle cycles, whose winding numbers vanish off  $D(c)$ , so the indicators of the two regular closed polygons agree off the closed set  $D(c)$ ;

and in edge coordinates a profile is the graph of a function of absolute value at most  $t$  over its chord, so every point of a triangle is within  $t$  of the chord and of the profile. The same holds for the complementary profiles and for placed copies.

**Proposition 2.5** (Straightening). *Replacing every profiled edge by its chord defines a bijection  $\mathcal{S}_c$  from the tilings of  $P_c$  onto the union of the two homochiral tiling spaces of  $H$ , preserving the isometry placing each tile. It is an MLD equivalence. For  $c, d \in U$  the map  $\mathcal{R}_{c,d} = \mathcal{S}_d^{-1}\mathcal{S}_c$ , which replaces each placed copy of  $P_c$  by the copy of  $P_d$  with the same placement, is an MLD equivalence between the tilings of  $P_c$  and those of  $P_d$ .*

*Proof.* For a placed tile  $X = gP_c$  let  $X' = gH$  be its straightening. As 1-chains,

$$\partial X - \partial X' = \sum_{j=0}^{13} \gamma_j(X), \quad (2.10)$$

where  $\gamma_j(X)$  is the closed profile cycle on edge  $j$ , bounding a triangle of diameter less than 2. Let a tiling  $\mathcal{T}$  by  $P_c$  be given; it is homochiral by Lemma 2.3, and after a reflection of the plane all placements are direct, so the boundaries of the  $X'$  are positively oriented. By Lemma 2.3 the profile cycles occur in pairs  $\gamma_j(X) = -\gamma_{j'}(Y)$ , one pair for each contact. For  $z$  off all profiled and straightened sides, only the finitely many cycles whose triangles contain  $z$  have nonzero winding number about  $z$ , and these cancel in pairs; summing (2.10) over  $\mathcal{T}$  gives

$$\sum_{X \in \mathcal{T}} \text{wind}(\partial X', z) = \sum_{X \in \mathcal{T}} \text{wind}(\partial X, z) = 1.$$

The family  $(X')$  is locally finite, and the equality extends by local constancy to every point off the straightened sides, so Lemma 2.4 shows that it is a tiling by  $H$ , homochiral by construction. Conversely, for a homochiral tiling by  $H$  the alignment property and the definition of the complementary path give the pairing of the profile cycles, and the same computation with (2.10) read from right to left shows that the profiled tiles form a tiling.

Both maps use the same placements, so they are inverse to each other and commute with isometries. By (2.9) an image tile meeting a unit disc has its source tile meeting the concentric disc of radius  $1+t$ ; replacing the source tiles meeting the larger disc and retaining the images meeting the unit disc gives the image patch, so both maps are local with radius  $63/50$ , and the same bound shows that the replaced family is locally finite. The placement of a tile is well defined because the symmetry groups are trivial: for  $P_c$  by Proposition 2.1, and for  $H$  because a symmetry preserves the side of length 2, a rotation would be the half-turn about its midpoint, a reflection fixing it pointwise exchanges the two sides of its supporting line, and a reflection exchanging its endpoints would exchange the adjacent vertices  $b_8$  and  $b_{12}$ , of angles  $270^\circ$  and  $90^\circ$ . The statement for  $\mathcal{R}_{c,d}$  follows by composition, with radius  $38/25$  since  $d_H(P_c, P_d) \leq t+v$  for  $d = u - iv \in U$ .  $\square$

### 3. THE HIERARCHY OF SUPERTILES

Write  $\mathcal{X}_c$  for the set of tilings by  $P_c$  and  $\mathcal{X}_H^\pm$  for the two handedness classes of homochiral tilings by  $H$ . Proposition 2.5 gives a placement-preserving bijection

$$\mathcal{S}_c : \mathcal{X}_c \longrightarrow \mathcal{X}_H^+ \sqcup \mathcal{X}_H^-. \quad (3.1)$$

The hierarchy of [21] enters in the following form.

**Theorem 3.1** (Smith, Myers, Kaplan and Goodman-Strauss). *The polygon  $H$  admits homochiral tilings of the plane, and every such tiling is non-periodic. Every homochiral tiling by  $H$  has a unique hierarchical structure: its tiles group into supertiles of nine marked types, each supertile of level  $m+1$  is a fixed union of supertiles of level  $m$ , the grouping at each level is determined by bounded neighbourhoods, and it commutes with translations.*

Existence and non-periodicity are [21, Theorem 2.2], and the hierarchy is [21, Theorems 4.1 and 5.1], with the marked Spectres of [21, Figure 5.3] and the key points of [21, Appendix A]. The zero-width segments by which [21, Figure 5.3] records markings are not part of the supports.

**Lemma 3.2** (Supports of supertiles). *At every positive level the supports of the supertiles of a homochiral tiling by  $H$  form a tiling combinatorially equivalent to a tiling by hexagons; in particular every support is a closed topological disc.*

*Proof.* Combinatorial equivalence has the meaning of [21, Section 2.2], a homeomorphism of tiling complexes. Under the equivalence of [21, Theorem 3.1] between  $H$ -tilings and hat-turtle tilings, the level-one clusters correspond to the clusters of [21, Theorem 4.1], whose tiling is homeomorphic to a tiling by marked hexagons, and the composition of [21, Theorem 5.1] groups these hexagons into patches homeomorphic to hexagons; pulling back and iterating gives the assertion at every level.  $\square$

**3.1. Transport of the hierarchy.** For a tiling  $\mathcal{T}$  let  $\text{Per}(\mathcal{T}) = \{v \in \mathbb{R}^2 : \mathcal{T} + v = \mathcal{T}\}$ , equality meaning equality of the collections of placed tiles.

**Proposition 3.3.** *For every  $c \in U$  the set  $\mathcal{X}_c$  is nonempty, and every tiling in it is homochiral and non-periodic. Each such tiling carries a unique hierarchy, obtained by transporting that of its straightening, and at every fixed level the supertile containing a tile, with its marking, is determined by a bounded neighbourhood of the tile. For  $c, d \in U$  the map  $\mathcal{R}_{c,d}$  preserves the hierarchy and satisfies  $\text{Per}(\mathcal{R}_{c,d}\mathcal{T}) = \text{Per}(\mathcal{T})$ .*

*Proof.* A tiling in  $\mathcal{X}_H^+$  exists by Theorem 3.1; replacing each placed copy of  $H$  by the copy of  $P_c$  with the same placement gives a tiling by  $P_c$ , by Proposition 2.5. Conversely every tiling  $\mathcal{T} \in \mathcal{X}_c$  straightens to a homochiral tiling by  $H$  with the same placements. For any translation  $\tau$  and any placed tile  $gP_c$  one has  $\mathcal{S}_c(\tau gP_c) = \tau gH = \tau \mathcal{S}_c(gP_c)$ , and applying this to all tiles and then the inverse replacement gives

$$\text{Per}(\mathcal{T}) = \text{Per}(\mathcal{S}_c\mathcal{T}) = 0. \quad (3.2)$$

Homochirality is Lemma 2.3. The hierarchy of  $\mathcal{S}_c\mathcal{T}$  partitions the placements into supertiles, and the same partition of the placements of  $\mathcal{T}$  defines its hierarchy: existence, disjointness of the groupings and their consistency across levels transfer, and two hierarchies on  $\mathcal{T}$  would give two on  $\mathcal{S}_c\mathcal{T}$ , contradicting Theorem 3.1. At a fixed level the grouping of  $\mathcal{S}_c\mathcal{T}$  is read in a bounded neighbourhood, and straightening is local, so the same holds in  $\mathcal{T}$ , with a bound depending on the level. The last assertion follows from (3.2) applied twice.  $\square$

*Proof of Theorem A.* Proposition 2.1 gives the first sentence, Lemma 2.3 the edge-to-edge matching and homochirality, Proposition 3.3 existence and non-periodicity, and Proposition 2.5 the two mutual local derivabilities.  $\square$

A tile  $X$  is a *Spectre* in the sense of [21, Section 2] if it admits only homochiral tilings, its tilings and the homochiral tilings of  $H$  correspond placement by placement in both directions, and corresponding tilings have the same tiling vertices, the points shared by three or more tiles.

**Corollary 3.4.** *For every  $c \in U$  the polygon  $P_c$  is a Spectre.*

*Proof.* The first three conditions are Theorem A. For the fourth, the interior of a profiled edge lies in exactly two tiles by Lemma 2.3, so a point shared by three or more tiles is an inherited vertex. The tiles incident to such a vertex keep it as a vertex, with the same angle, under straightening, and their angles sum to  $2\pi$ , so no further tile contains the point; the same argument for the inverse replacement, using the alignment property, shows that the incident tiles coincide, and hence the points shared by three or more tiles.  $\square$

**3.2. Supertiles in coordinates.** The nine marked types are named, as in [21],

$$\Gamma, \Delta, \Theta, \Lambda, \Xi, \Pi, \Sigma, \Phi, \Psi. \quad (3.3)$$

At level zero the eight types other than  $\Gamma$  have support  $H$  with their respective markings, and  $\Gamma$  has the two-tile support

$$H_\Gamma = H \cup (\omega H + 3 - \omega + 2\omega^3), \quad (3.4)$$

the Mystic of [21]. A one-tile support is called a *singleton*. Supertiles of level  $m$  are obtained from those of level  $m - 1$  by the child table of Section 4 (Table 1): an ordinary type has eight children in slots  $0, \dots, 7$  and  $\Gamma$  has seven, slot 2 being absent. Each supertile carries four *key points*, which at level zero are

$$k_0 = (b_3, b_5, b_7, b_{11})^t. \quad (3.5)$$

The placements of the children and the key points of the parent are given by the recurrences of Section 4; the next proposition identifies that coordinate description with the published one.

**Proposition 3.5.** *The child table and the placements of Section 4 realize, at every level, the marked refinement of Theorem 3.1.*

*Proof.* Before the final reflection, the rotations of slots  $0, \dots, 7$  relative to the parent frame are

$$(r_0, \dots, r_7) = (1, \zeta, \zeta, \zeta^2, \zeta^3, \zeta^3, \zeta^4, \zeta^2), \quad \zeta = \omega^2,$$

and, with the key points of a slot numbered  $0, 1, 2, 3$ , consecutive slots are joined by key  $u_\sigma$  of slot  $\sigma - 1$  meeting key  $v_\sigma$  of slot  $\sigma$ , where

$$(u_\sigma, v_\sigma)_{\sigma=1}^7 = ((3, 1), (2, 0), (3, 1), (3, 1), (2, 0), (3, 1), (3, 3)). \quad (3.6)$$

These are the data of the four-key-point construction [21, Appendix A, Figure A.1] in the slot numbering of Table 1; the same joins and the same child table are used in the authors' implementation.<sup>2</sup> If slot  $\sigma$  is placed by  $z \mapsto r_\sigma z + t_\sigma$  with  $t_0 = 0$ , each join fixes the next translation,

$$t_\sigma = t_{\sigma-1} + r_{\sigma-1}k_{u_\sigma} - r_\sigma k_{v_\sigma}, \quad (3.7)$$

so the seven joins determine all placements; reflection in the imaginary axis then gives  $z \mapsto -\overline{r_\sigma z + t_\sigma}$ . The key points of the parent are, in order, key 2 of slot 6, key 1 of slot 5, key 2 of slot 3 and key 1 of slot 0. The Mystic cluster omits slot 2 from this construction.

At level zero the vertices of  $H$ , the key points (3.5) and the compound (3.4) agree with the marked realization. If the agreement holds at level  $m$ , then (3.7) forces the placements at level  $m + 1$  to agree, the child table assigns the same markings, and the selection of parent keys gives the same key vector. Induction completes the proof.  $\square$

One application of the marked rule reverses the orientation of the reference frame in which the next level is read: the child placements are antilinear in the parent frame, and two consecutive applications are complex affine. The tiles inside a supertile all have one handedness; only the frame is reversed. This reversal is the source of the signs in (1.1) and of the orientation character in Theorem D.

**3.3. Changing the profile.** For a finite collection  $\mathcal{P}$  of tiles of a homochiral tiling by  $H$ , a finite union of supertiles for instance, write  $|\mathcal{P}|$  for its support and  $|\mathcal{P}^c|$  for the support of the  $P_c$ -tiles with the same placements; the supertiles of  $P_c$  are these patches, with the markings and the key points of their  $H$ -supertiles.

**Lemma 3.6** (Profiles on a patch). *Let  $c = x - it \in U$  and let  $\mathcal{P}$  be a finite nonempty patch of tiles of  $H$ ; call a marked edge of a tile of  $\mathcal{P}$  exposed if its matching tile lies outside  $\mathcal{P}$ .*

<sup>2</sup><https://cs.uwaterloo.ca/~csk/spectre/spectre.js>, accessed 27 September 2026.

- (i) *The frontier of  $|\mathcal{P}|$  is the union of the exposed edges and that of  $|\mathcal{P}^c|$  the union of their profiles; the oriented boundary chain of  $|\mathcal{P}^c|$  is obtained from that of  $|\mathcal{P}|$  by replacing every exposed unit edge with its profiled edge.*
- (ii) *The Hausdorff distance between  $\partial|\mathcal{P}|$  and  $\partial|\mathcal{P}^c|$  is at most  $t \leq 13/50$ , and the two supports agree outside the closed  $t$ -neighbourhood of  $\partial|\mathcal{P}|$ .*

*Proof.* (i) By the alignment property an edge that is not exposed is shared by two tiles of  $\mathcal{P}$ , which meet along it with opposite traversal, and their profiles on it coincide by the definition of the complementary path; so the two chords cancel in the boundary chain of  $|\mathcal{P}|$  and the two profiled edges in that of  $|\mathcal{P}^c|$ . An exposed edge lies in the frontier, endpoints included, since the outside tile meeting it is the closure of its interior; conversely, at a frontier point of  $|\mathcal{P}|$ , or of  $|\mathcal{P}^c|$  with profiles in place of chords, the incident tiles partition a small disc into sectors, some in  $\mathcal{P}$  and some not, and a transition between the two kinds lies on an exposed edge through the point; neither step uses connectedness of the patch or the absence of holes. (ii) In edge coordinates the profile and its chord have points with the same first coordinate and second coordinates differing by at most  $t$ , which with (i) gives the Hausdorff bound. Subtracting the two boundary chains leaves the boundaries of the exposed profile triangles, so off their union and the tile boundaries the winding numbers of the two chains agree and are the indicators of the two supports, the tiles having disjoint interiors; both supports are regular closed, and the argument of (2.9) extends the agreement to every point off the closed union of the triangles, each of which lies within distance  $t$  of its chord. On  $U$ ,  $t \leq 13/50$ .  $\square$

#### 4. THE BOUNDARY OF A SUPERTILE AND ITS LIMIT CURVE

##### 4.1. The placements of the children. Put

$$\omega = e^{\pi i/6}, \quad \zeta = \omega^2, \quad \lambda = 4 + \sqrt{15}, \quad \theta = \phi^3 = 2 + \sqrt{5}.$$

All coordinates in this section lie in the number field  $\mathcal{F} = \mathbb{Q}(\zeta_{12}, \sqrt{5}) = \mathbb{Q}(\sqrt{3}, \sqrt{5}, i)$ . A bar denotes complex conjugation;  $\tau$  denotes the automorphism of  $\mathcal{F}$  fixing  $\omega$  and sending  $\sqrt{5}$  to  $-\sqrt{5}$ , so that  $\tau(\lambda) = \lambda^{-1}$ .

The key points (3.5) of the level-zero singleton are

$$k_0 = \kappa \mathbf{1} + d, \quad \kappa = 1 - \omega + \omega^2 + 2\omega^3, \quad d = \begin{pmatrix} 1 & 2 & -2 & -2 \\ 2 & 2 & -2 & -1 \\ 2 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix}.$$

Number the child slots  $0, \dots, 7$ , let  $e_0, \dots, e_3$  be the standard row vectors, and define

$$\begin{aligned} r_\sigma &= \zeta^{a_\sigma}, & (a_\sigma)_{\sigma=0}^7 &= (0, 1, 1, 2, 3, 3, 4, 2), \\ \ell_0 &= 0, & \ell_\sigma &= \ell_{\sigma-1} + r_{\sigma-1}e_{u_\sigma} - r_\sigma e_{v_\sigma} \quad (1 \leq \sigma \leq 7), \end{aligned} \tag{4.1}$$

with the joins  $(u_\sigma, v_\sigma)$  of (3.6). The children of each type are listed in Table 1.

The key points of consecutive levels and the placement of the child in slot  $\sigma$  inside a parent of level  $m$  are

$$k_{m+1} = -\overline{\mathbf{E}k_m}, \quad f_{m,\sigma}(z) = \overline{-r_\sigma z + \ell_\sigma k_m}, \quad \mathbf{E} = \begin{pmatrix} 1-\zeta & 2-\zeta & -1 & 2\zeta-1 \\ 1-\zeta & 1-2\zeta & \zeta-1 & 2\zeta \\ -\zeta & 1-2\zeta & 2\zeta-1 & 1+\zeta \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

and we put  $\mathbf{R} = \overline{\mathbf{E}}\mathbf{E}$ . The conjugations in (4.2) are the reversal of the reference frame described after Proposition 3.5; they do not introduce reflected tiles. Two consecutive placements compose to the complex affine maps

$$f_{m+1,\sigma}f_{m,\sigma'}(z) = r_{\sigma,\sigma'}z + \eta_{\sigma,\sigma'}k_m, \quad r_{\sigma,\sigma'} = \overline{r_\sigma}r_{\sigma'}, \quad \eta_{\sigma,\sigma'} = \overline{r_\sigma}\ell_{\sigma'} + \overline{\ell_\sigma}\mathbf{E}, \quad k_{m+2} = \mathbf{R}k_m. \tag{4.3}$$

| type      | 0      | 1        | 2      | 3        | 4        | 5      | 6         | 7        |
|-----------|--------|----------|--------|----------|----------|--------|-----------|----------|
| $\Gamma$  | $\Pi$  | $\Delta$ | $-$    | $\Theta$ | $\Sigma$ | $\Xi$  | $\Phi$    | $\Gamma$ |
| $\Delta$  | $\Xi$  | $\Delta$ | $\Xi$  | $\Phi$   | $\Sigma$ | $\Pi$  | $\Phi$    | $\Gamma$ |
| $\Theta$  | $\Psi$ | $\Delta$ | $\Pi$  | $\Phi$   | $\Sigma$ | $\Pi$  | $\Phi$    | $\Gamma$ |
| $\Lambda$ | $\Psi$ | $\Delta$ | $\Xi$  | $\Phi$   | $\Sigma$ | $\Pi$  | $\Phi$    | $\Gamma$ |
| $\Xi$     | $\Psi$ | $\Delta$ | $\Pi$  | $\Phi$   | $\Sigma$ | $\Psi$ | $\Phi$    | $\Gamma$ |
| $\Pi$     | $\Psi$ | $\Delta$ | $\Xi$  | $\Phi$   | $\Sigma$ | $\Psi$ | $\Phi$    | $\Gamma$ |
| $\Sigma$  | $\Xi$  | $\Delta$ | $\Xi$  | $\Phi$   | $\Sigma$ | $\Pi$  | $\Lambda$ | $\Gamma$ |
| $\Phi$    | $\Psi$ | $\Delta$ | $\Psi$ | $\Phi$   | $\Sigma$ | $\Pi$  | $\Phi$    | $\Gamma$ |
| $\Psi$    | $\Psi$ | $\Delta$ | $\Psi$ | $\Phi$   | $\Sigma$ | $\Psi$ | $\Phi$    | $\Gamma$ |

TABLE 1. The nine marked types and the types of their children by slot;  $\Gamma$  has no child in slot 2.

By Proposition 3.5 these formulas are the marked refinement of Theorem 3.1 in coordinates.

**Lemma 4.1** (Eigenvector decomposition of the key points). *Put  $\xi = (R - \lambda^{-1}I)d/(\lambda - \lambda^{-1})$  and  $\xi' = \tau\xi$ . Then*

$$k_{2m} = \kappa \mathbf{1} + \lambda^m \xi + \lambda^{-m} \xi'.$$

*There is  $c_* \in \mathcal{F}$  with  $-\overline{E\xi} = c_*\xi$  and  $|c_*|^2 = \lambda$ , and the child placements converge, after subtraction of the key points and division by the expanding eigencomponent, to the antilinear contractions*

$$g_\sigma(z) = \mu_\sigma \bar{z} + t_\sigma, \quad \mu_\sigma = -\frac{\bar{r}_\sigma}{c_*}, \quad t_\sigma = -\frac{\bar{\ell}_\sigma \xi}{c_*}, \quad (4.4)$$

*of ratio  $\lambda^{-1/2}$ , with  $g_\sigma g_{\sigma'}(z) = \lambda^{-1}(r_{\sigma,\sigma'}z + \eta_{\sigma,\sigma'}\xi)$ .*

*Proof.* Reduction in  $\mathbb{Q}(\omega)$  with  $\omega^4 = \omega^2 - 1$  gives

$$Rd = \begin{pmatrix} 8 & 11 & -19 & -14 \\ 17 & 15 & -17 & -9 \\ 15 & 10 & -6 & 0 \\ -4 & -1 & -3 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix}, \quad R^2d = 8Rd - d, \quad R\mathbf{1} = \mathbf{1}.$$

Thus  $d$  lies in the sum of the eigenspaces of  $R$  for the eigenvalues  $\lambda$  and  $\lambda^{-1}$ , which  $\tau$  exchanges, and  $\xi, \xi'$  are its two eigencomponents; the formula for  $k_{2m}$  follows from  $k_{2m} = R^m k_0$  and  $R\mathbf{1} = \mathbf{1}$ . Substituting the coordinates gives  $-\overline{E\xi} = c_*\xi$ , and applying this twice gives  $|c_*|^2 \xi = R\xi = \lambda\xi$ , whence  $|c_*|^2 = \lambda$  since  $\xi \neq 0$ . The formulas for  $g_\sigma$  and  $g_\sigma g_{\sigma'}$  follow from (4.2) and (4.3) by subtracting the constant key and dividing by the expanding factor. Every identity here is an equality in  $\mathcal{F}$ , verified coefficientwise as described in Appendix B.  $\square$

**4.2. The ordered subdivision.** Boundary substitutions of inflation tilings are standard [1, Ch. 6], [7]; the growth matrix does not record their order and signs. Name the arcs of the level-zero singleton  $A, B, C, D$ , running between the consecutive key points  $k_{0,0} \rightarrow k_{0,1} \rightarrow k_{0,2} \rightarrow k_{0,3} \rightarrow k_{0,0}$ . A pair  $(\sigma, j)$  denotes the  $j$ -th arc of the child in slot  $\sigma$ , traversed backwards. The complete ordered data of the subdivision are those of Table 2.

| arc | children                                                 |
|-----|----------------------------------------------------------|
| $A$ | $(6, 2), (5, 3), (5, 2)$                                 |
| $B$ | $(5, 1), (4, 2), (3, 3)$                                 |
| $C$ | $(3, 2), (2, 3), (2, 2), (2, 1), (1, 2), (0, 3), (0, 2)$ |
| $D$ | $(0, 1), (7, 4), (6, 3)$                                 |

TABLE 2. The children of the four arcs, as pairs  $(\sigma, j)$  in order, each traversed backwards.

For the type  $\Gamma$  the third word is replaced by

$$(3, 2), (2, 4)^+, (1, 2), (0, 3), (0, 2), \quad (4.5)$$

the superscript marking forward traversal; this accounts for the absent child in slot 2.

**Proposition 4.2** (Subdivision). *For every  $m \geq 1$  the exposed boundary of the canonical singleton supertile of level  $m$  is subdivided according to Table 2, with the placements (4.2). Forgetting the slot numbers and keeping the traversal directions gives the rule (1.1). The limit arcs of Theorem 4.6 have the same order and the same endpoints.*

*Proof.* Tracing the eight placements and the child table at level one gives Table 2, and omitting slot 2 gives (4.5); this finite calculation is Proposition B.3. To pass to every level one needs that the recursively defined words are the exposed boundaries. Appendix B.2 proves three ordered path identities, valid at every level:  $B$  is a rotated copy of  $A$  in reversed traversal;  $C$  is a rotated copy of  $A$  followed by a centrally symmetric remainder; and  $D$  satisfies a free-path relation with  $A$  and  $C$ . These identities give the spherical diagrams of Appendix B.3 for the two parent clusters, and Proposition B.12 concludes by induction on the level.

For a reversed child of type  $j$  in slot  $\sigma$  the endpoints before the common conjugation have the row expressions  $r_\sigma e_j + \ell_\sigma$  and  $r_\sigma e_{j-1} + \ell_\sigma$ , indices modulo four; consecutive endpoints agree coefficientwise except at the first join of the word of  $D$ , where the difference is a multiple of  $\nu = (1, \zeta - 2, 1 - \zeta, 0)$ , and  $\nu k_m = 0$  for all  $m$  since  $\nu k_0 = 0$  and  $\nu \bar{E} = (\zeta - 1)\bar{\nu}$ . The extreme endpoints of each word are the key points of the parent, and the identities pass to the rescaled limit.  $\square$

**4.3. Bounding polygons.** The ordered words do not by themselves give an embedded limit. Write  $g_{\sigma_1 \dots \sigma_r} = g_{\sigma_1} \dots g_{\sigma_r}$  and let

$$p_j = \frac{t_j + \mu_j \bar{t}_j}{1 - \lambda^{-1}}, \quad j \in \{2, 4, 7\},$$

be the fixed point of  $g_j$ . Denote the limit key points by  $x_0, \dots, x_3$  and let  $\mathcal{C}_1, \dots, \mathcal{C}_4$  be the convex polygons with the counterclockwise vertex lists

$$\begin{aligned} \mathcal{C}_1 &= (g_{53}p_2, x_1, g_6p_4, g_{6456}p_4, x_0, g_{523}p_2, g_5p_2), \\ \mathcal{C}_2 &= (x_1, g_3p_2, g_{3223}p_2, x_2, g_{456}p_4, p_4, g_{56}p_4), \\ \mathcal{C}_3 &= (g_{03}p_2, x_3, g_{3456}p_4, x_2, g_{223}p_2, p_2, g_{23}p_2, g_0p_2), \\ \mathcal{C}_4 &= (g_6p_2, g_{6223}p_2, x_0, g_{706}p_4, p_7, g_{06}p_4, x_3, g_{053}p_2). \end{aligned} \quad (4.6)$$

**Lemma 4.3** (Nesting and separation). *The image of the polygon of each child in Table 2 lies in the polygon of its parent. The polygons of consecutive children of one parent meet exactly in their common endpoint, and those of nonconsecutive children are disjoint. An endpoint of a parent arc lies only in the polygon of its first or last child, respectively. The four polygons  $\mathcal{C}_i$  meet exactly in the cyclic common endpoints.*

*Proof.* For  $a, b, z \in \mathbb{C}$  put  $[a, b, z] = \text{Im}(\overline{(b-a)}(z-a))$ ; a counterclockwise convex polygon is the intersection of the half-planes  $[a, b, z] \geq 0$  over its consecutive vertex pairs. The claims are inequalities and equalities among finitely many such determinants at the points (4.6) and their images under (4.4), all of which lie in  $\mathcal{F}$ ; they are listed and decided exactly in Lemma B.2, and convexity extends the vertex tests to the polygons.  $\square$

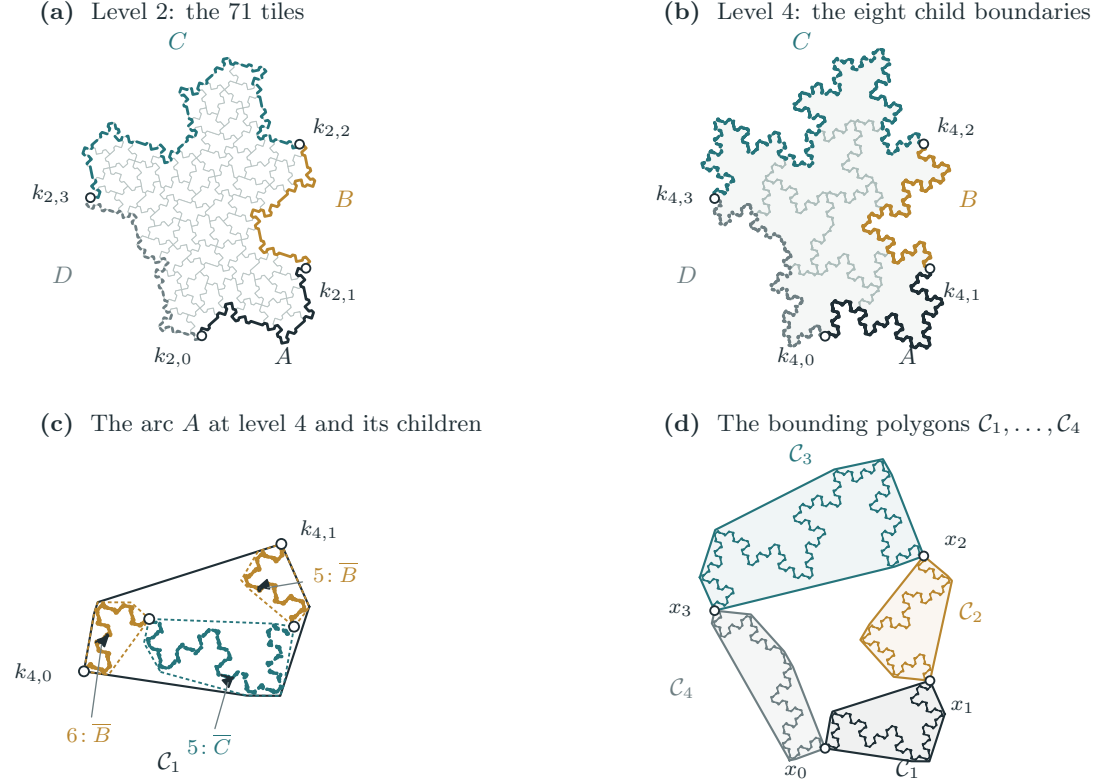


FIGURE 2. Supertile boundaries. (a), (b) Levels 2 and 4, rescaled by  $z \mapsto \lambda^{-m}(z - \kappa)$  at level  $2m$ , with the arcs  $A, B, C, D$  and the key points  $k_{2m,j}$ ; (a) shows the 71 tiles, (b) the eight child boundaries. (c) At twice the scale, the arc  $A$  of level 4 with its children  $(6, 2), (5, 3), (5, 2)$ , traversed backwards, inside  $C_1$  and, dashed, the polygons  $g_\sigma(C_j)$ . (d) The bounding polygons  $C_1, \dots, C_4$  with the limit key points  $x_j$  and the level-4 boundary.

**4.4. The limit curve and its dimension.** The unsigned incidence matrix of Table 2 and its Perron eigenvector are

$$W = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 4 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \quad Ww = \theta w, \quad w = (1, 1, \sqrt{5}, 1)^t. \quad (4.7)$$

Let  $\mathcal{G}$  be the directed graph with vertex set  $\{1, 2, 3, 4\}$ , the arcs  $A, B, C, D$ , and with one edge  $i \rightarrow j$  for each entry  $(\sigma, j)$  of the word of  $i$ , labelled by the antilinear similarity  $g_\sigma$  of ratio  $r = \lambda^{-1/2}$ ;  $W$  counts its edges. This is a graph-directed construction in the sense of [15]. Write  $x_0, \dots, x_3$  for the limit key points, indices modulo 4, and give the entry  $(\sigma, j)$  of the word of  $i$  the parameter interval of length  $w_j/(\theta w_i)$ ; by (4.7) these lengths sum to one.

**Lemma 4.4** (The limit arcs). *There is a unique quadruple of continuous paths  $\kappa_i : [0, 1] \rightarrow \mathbb{C}$  with  $\kappa_i(0) = x_{i-1}$ ,  $\kappa_i(1) = x_i$ , such that  $\kappa_i$  is the concatenation, over the entries  $(\sigma, j)$  of the word of  $i$  in order and on their parameter intervals, of the reversed paths  $g_\sigma \circ \kappa_j \circ (1 - \cdot)$ . The sets  $K_i = \kappa_i([0, 1])$  are the unique nonempty compact sets with  $K_i = \bigcup_{(\sigma, j)} g_\sigma(K_j)$ . The boundary arcs of the singleton supertile of level  $2m$ , rescaled by  $z \mapsto \lambda^{-m}(z - \kappa)$  and parametrized in the same way, converge uniformly to the  $\kappa_i$  at rate  $O(\lambda^{-m})$ .*

*Proof.* Let  $\mathcal{C}_i$  be the space of continuous paths from  $x_{i-1}$  to  $x_i$  with the uniform metric. By the endpoint identities of Proposition 4.2 the concatenation described in the statement is a map  $\Phi : \prod_i \mathcal{C}_i \rightarrow \prod_i \mathcal{C}_i$ , and since each  $g_\sigma$  is a similarity of ratio  $r < 1$ ,  $\Phi$  contracts the uniform metric by  $r$ ; its unique fixed point is the quadruple  $(\kappa_i)$ . The images satisfy the set equations, and a family of nonempty compact sets satisfying them is unique by the contraction principle for the Hausdorff metric [11, 15].

Put  $\Psi = \Phi^2$ , which has the same fixed point and contraction ratio  $\lambda^{-1}$ , and let  $\Psi_m$  be the analogous operator built from the two-level placements (4.3) rescaled at level  $2m$ . By Lemma 4.1 the rescaled key vector is  $\xi + \lambda^{-2m}\xi'$ , and since  $\mathbf{E}\mathbf{1} = \mathbf{1}$  and  $\ell_\sigma \mathbf{1} = 1 - r_\sigma$ , the rescaled two-level map differs from  $g_\sigma g_{\sigma'}$  by the translation  $\lambda^{-1}\eta_{\sigma,\sigma'}\xi'\lambda^{-2m}$ , so  $\|\Psi_m - \Psi\|_\infty \leq C\lambda^{-2m}$  on bounded sets, the corresponding pieces being compared in the supremum norm of the plane since the finite paths have moving endpoints. The rescaled boundary arcs  $\beta_m$  of level  $2m$  satisfy  $\beta_{m+1} = \Psi_m(\beta_m)$  by Proposition 4.2, and  $E_m = \|\beta_m - \kappa\|_\infty$  obeys  $E_{m+1} \leq \lambda^{-1}E_m + C\lambda^{-2m}$ , whence  $E_m = O(\lambda^{-m})$ .  $\square$

**Lemma 4.5** (Jordan curve). *Each  $\kappa_i$  is injective,  $K_i \cap K_{i+1} = \{x_i\}$ ,  $K_i \cap K_{i+2} = \emptyset$ , and  $K = K_1 \cup K_2 \cup K_3 \cup K_4$  is a Jordan curve.*

*Proof.* By Lemma 4.3,  $\kappa_i([0, 1]) \subset \mathcal{C}_i$  for the fixed point of  $\Phi$ , since  $\Phi$  preserves the closed set of quadruples with  $\kappa_i([0, 1]) \subset \mathcal{C}_i$ . An endpoint of  $\kappa_i$  lies only in the polygon of the first or last entry, and by induction only in the nested first or last cylinders, so its unique preimage is 0 or 1. A cylinder of depth  $n$  beginning at type  $i$  and ending at type  $j$  has parameter length  $w_j/(\theta^n w_i) \leq \sqrt{5}\theta^{-n}$ , the intermediate weights telescoping, so two parameters  $u \neq u'$  in  $[0, 1]$  lie in different cylinders at some depth; the polygons of these cylinders are disjoint unless the cylinders are consecutive, in which case they meet only in the common endpoint, whose parameter is unique. Hence  $\kappa_i(u) \neq \kappa_i(u')$ . The intersections of the  $\mathcal{C}_i$  give the intersections of the  $K_i$ , and the four arcs form a Jordan curve.  $\square$

**Theorem 4.6** (The limit curve). *Rescaled by  $z \mapsto \lambda^{-m}(z - \kappa)$ , the boundaries of the canonical singleton supertiles of level  $2m$  converge in Hausdorff distance at rate  $O(\lambda^{-m})$  to the Jordan curve  $K$ , whose Hausdorff and box dimensions are*

$$\delta = \frac{\log \theta^2}{\log \lambda} = \frac{6 \log \phi}{\log(4 + \sqrt{15})}, \quad 0 < \mathcal{H}^\delta(K) < \infty.$$

*For a placed singleton supertile  $Q$  of level  $2m$  let  $F_Q$  be the isometry carrying the canonical supertile to the straightened placement of  $Q$ . The boundary of  $Q$ , rescaled by  $z \mapsto \lambda^{-m}(F_Q^{-1}(z) - \kappa)$ , has the same limit and rate, with a constant uniform over  $c \in U$ , the tiling and  $Q$ .*

*Proof.* Convergence is Lemma 4.4. Its recurrence gives, with  $a = \lambda^{-1}$  and with  $C_1$  the maximum of  $|\lambda^{-1}\eta_{\sigma,\sigma'}\xi'|$  over the slots,

$$E_m \leq a^m E_0 + C_1 \sum_{j < m} a^{m-1-j} a^{2j} \leq \left(E_0 + \frac{C_1}{a(1-a)}\right) a^m.$$

In the frame  $F_Q$  every singleton has the canonical unprofiled boundary, and profiling moves it by at most  $13/50$  before rescaling, by Lemma 3.6(ii), whether or not  $F_Q$  reverses orientation; so one constant serves for all  $c$ , tilings and  $Q$ . The open sets  $\text{int } \mathcal{C}_i$  satisfy the open set condition for  $\mathcal{G}$ : by Lemma 4.3,  $g_\sigma(\text{int } \mathcal{C}_j) \subset \text{int } \mathcal{C}_i$  for every entry  $(\sigma, j)$  of the word of  $i$ , and the images for distinct entries are disjoint, the polygons of consecutive entries meeting only in a point. The strongly connected components of  $\mathcal{G}$  are  $\{1, 2, 3\}$  and  $\{4\}$ , and the block of  $W$  on the first has Perron eigenvalue  $\theta$  with eigenvector  $(1, 1, \sqrt{5})$ . By the dimension theorem for graph-directed constructions [15],  $\dim_H K_i = \delta$  for  $i \leq 3$ , where  $r^\delta \theta = 1$ , and  $0 < \mathcal{H}^\delta(K_i) < \infty$ . For  $K_4$ ,  $g_0(K_1) \subset K_4$  gives  $\mathcal{H}^\delta(K_4) > 0$ , and  $K_4 = \{p_7\} \cup \bigcup_{n \geq 0} g_7^n(g_0 K_1 \cup g_6 K_3)$  gives  $\mathcal{H}^\delta(K_4) \leq \sum_n r^{n\delta}(\mathcal{H}^\delta(K_1) + \mathcal{H}^\delta(K_3)) < \infty$ . Finally  $K$  is covered by the  $O(\theta^j)$  cylinders of depth  $j$ , each of diameter at most  $\max_i \text{diam}(\mathcal{C}_i) r^j$ , so its upper box dimension is at most  $\delta$ , and the box dimension equals  $\delta$ .  $\square$

#### 4.5. Exposed edge counts.

**Proposition 4.7.** *The singleton supertile of level  $m$  has  $N_m = 4 + 2F_{3m+5}$  exposed profiled edges, distributed over its four arcs as  $(F_{3m+3}, F_{3m+3}, L_{3m+3}, F_{3m+3} + 4)$ .*

*Proof.* The counts evolve by the matrix  $W$  of (4.7), reversal being irrelevant to a count, and the four arcs of level zero have  $(2, 2, 4, 6)$  edges. From  $F_{r+3} = 2F_r + L_r$  and  $L_{r+3} = 5F_r + 2L_r$  one obtains by induction  $W^m(2, 2, 4, 6)^t = (F_{3m+3}, F_{3m+3}, L_{3m+3}, F_{3m+3} + 4)^t$ , and the sum is  $3F_{3m+3} + L_{3m+3} + 4 = 2F_{3m+5} + 4$ .  $\square$

*Proof of Theorem B.* Proposition 4.2 gives the subdivision at every level, and Theorem 4.6 gives the limit curve, its dimension, the finiteness and positivity of its  $\delta$ -dimensional measure, and the independence of  $c \in U$ .  $\square$

### 5. THE SUBDIVISION AS A FREE-GROUP AUTOMORPHISM

**5.1. The endpoint graph.** The arcs of the level-zero singleton run  $A : b_3 \rightarrow b_5$ ,  $B : b_5 \rightarrow b_7$ ,  $C : b_7 \rightarrow b_{11}$  and  $D : b_{11} \rightarrow b_3$ . Identify

$$b_3 \sim b_7 = v, \quad b_5 \sim b_{11} = w.$$

The result is a graph  $G$  with two vertices and four oriented edges,  $A, C : v \rightarrow w$  and  $B, D : w \rightarrow v$  (Figure 3). In the rule (1.1) each word has the same initial and terminal vertex as the edge it replaces, so the rule defines a graph map  $f : G \rightarrow G$  fixing  $v$  and  $w$ , in which an edge is mapped over the edges of its word in order, with reversed traversal where a bar appears. The subgraph  $\Theta$  spanned by  $A, B, C$  is a theta graph and is invariant under  $f$ . The graph  $G$  retains from the subdivision only the order, the direction and the endpoint type of each child arc, and none of its Euclidean placement.

The map  $r : G \rightarrow \Theta$  that fixes  $A, B, C$  and both vertices and sends  $D$  to  $B$ , preserving the parameter, is a retraction, and it commutes with  $f$ : on  $D$ ,

$$rf(D) = r(\overline{A}\overline{D}\overline{C}) = \overline{A}\overline{B}\overline{C} = f(B) = fr(D),$$

and  $r$  is the identity on the other edges; with each edge parametrized by  $[0, 1]$  and the letters of a word traversed on equal subintervals,  $rf = fr$  holds pointwise. The inclusion  $\Theta \hookrightarrow G$  is an  $f$ -equivariant section of  $r$ .

**5.2. The Fibonacci factorization.** Take  $v$  as base point and  $B$  as maximal tree. Then

$$\pi_1(\Theta, v) = F(a, c), \quad \pi_1(G, v) = F(a, c, e), \quad a = AB, \quad c = CB, \quad e = B^{-1}D.$$

**Lemma 5.1.** *The map induced by  $f$  on  $F(a, c)$  is the automorphism*

$$\rho(a) = c^{-1}a^{-1}c^{-1}, \quad \rho(c) = c^{-1}a^{-1}c^{-1}a^{-1}c^{-1},$$

*and on  $F(a, c, e)$  it is the extension of  $\rho$  by  $e \mapsto ce^{-1}c^{-1}$ .*

*Proof.* Concatenating the words of  $A$  and  $B$  gives

$$f(AB) = B^{-1}C^{-1}B^{-1}A^{-1}B^{-1}C^{-1} = (B^{-1}C^{-1})(B^{-1}A^{-1})(B^{-1}C^{-1}) = c^{-1}a^{-1}c^{-1},$$

and the words of  $C$  and  $B$  give  $\rho(c)$  in the same way. For  $e$ ,

$$f(B^{-1}D) = (CBA)(A^{-1}D^{-1}C^{-1}) = CBD^{-1}C^{-1} = ce^{-1}c^{-1}.$$

That  $\rho$  is an automorphism follows from Theorem 5.2 below; the extension is an automorphism because it inverts and conjugates the remaining generator by an element of the invariant factor  $F(a, c)$ .  $\square$

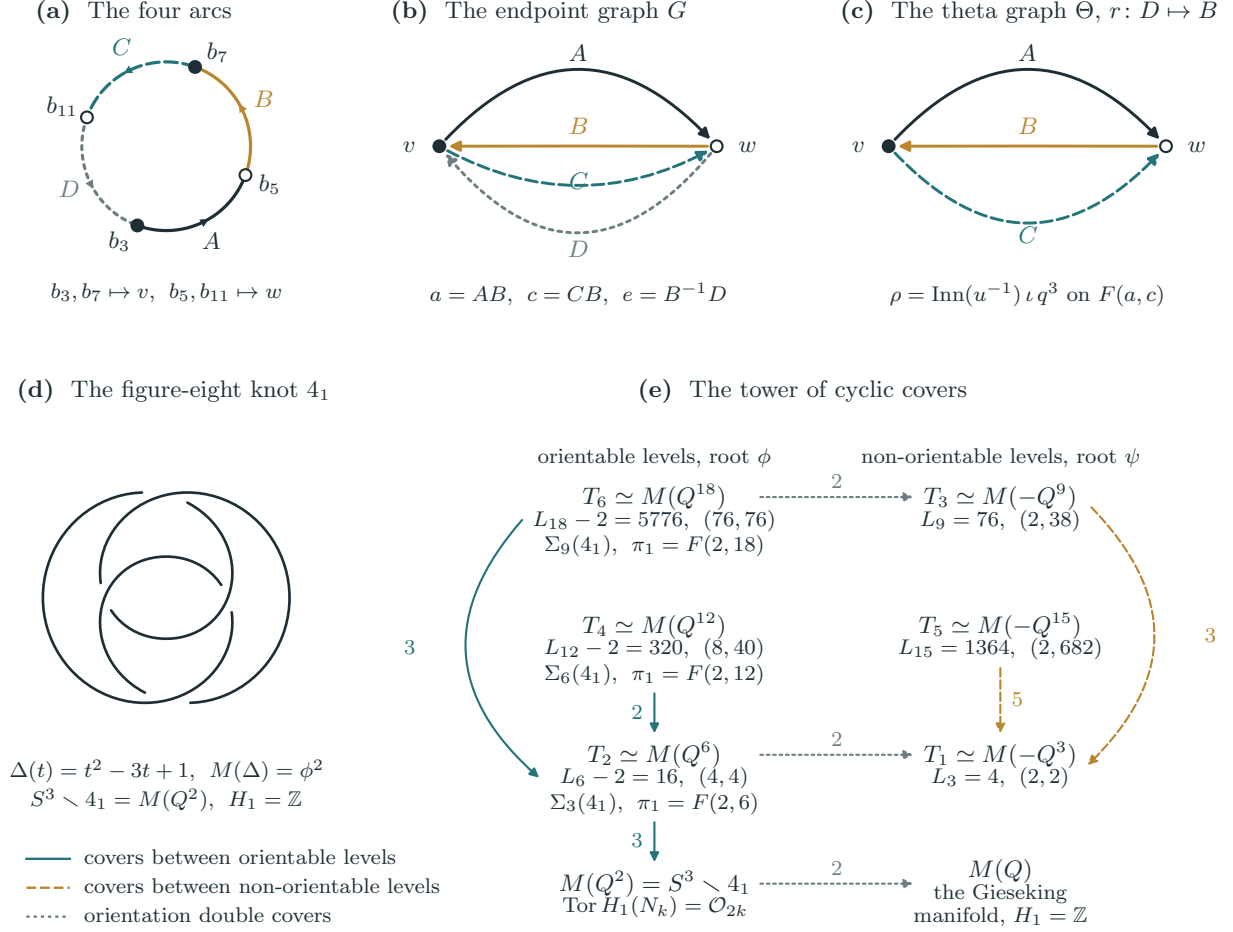


FIGURE 3. From the boundary cycle to the tower. (a) The four arcs and the identification  $b_3 \sim b_7 = v, b_5 \sim b_{11} = w$ . (b) The endpoint graph  $G$ . (c) The theta graph  $\Theta$  and the retraction  $r$ . (d) The figure-eight knot; its complement is  $M(Q^2)$ , and its Alexander polynomial  $\Delta$  enters Theorem 7.9. (e) The tower: orientable levels over  $M(Q^2)$ , non-orientable levels over the Gieseking manifold, the covering maps of prime degree with their degrees, and under each level the torsion order with its invariant factors and, for even  $n$ , the filling  $\Sigma_{3m}(4_1)$  with  $\pi_1 = F(2, 6m)$ .

Let  $q$  be the Fibonacci automorphism and  $\iota$  the inversion of generators,

$$q(a) = c, \quad q(c) = ac; \quad \iota(a) = a^{-1}, \quad \iota(c) = c^{-1};$$

$q$  is an automorphism with inverse  $a \mapsto ca^{-1}, c \mapsto a$ . Write  $\text{Inn}(g)$  for  $x \mapsto gxg^{-1}$ , and put

$$\sigma = \iota\rho, \quad u = cac.$$

**Theorem 5.2** (Restatement of Theorem C).  $\sigma(a) = cac, \sigma(c) = cacac$ , and

$$\sigma = \text{Inn}(u) \circ q^3, \quad \rho = \text{Inn}(u^{-1}) \circ \iota \circ q^3, \quad \iota\sigma = \sigma\iota, \quad \rho^2 = \sigma^2 = \text{Inn}(uq^3(u)) \circ q^6.$$

Consequently  $\rho$  is an automorphism, and its class in  $\text{Out } F(a, c) = \text{GL}_2(\mathbb{Z})$  is  $-Q^3$ .

*Proof.* The formulas for  $\sigma$  are read off from Lemma 5.1. Next,  $q^2(a) = ac, q^2(c) = cac, q^3(a) = cac$  and  $q^3(c) = accac$ ; conjugation by  $u = cac$  fixes  $cac$  and sends  $accac$  to  $cac \cdot accac \cdot c^{-1}a^{-1}c^{-1} =$

$cacac$ , which proves  $\sigma = \text{Inn}(u) \circ q^3$ . Applying  $\iota$  to both sides of  $\sigma = \text{Inn}(u) \circ q^3$  and using  $\iota \circ \text{Inn}(g) = \text{Inn}(\iota g) \circ \iota$  gives  $\rho = \iota\sigma = \text{Inn}(u^{-1}) \circ \iota \circ q^3$ . The two words  $\sigma(a), \sigma(c)$  are palindromes, so  $\iota\sigma(x) = \sigma(x)^{-1} = \sigma(x^{-1}) = \sigma\iota(x)$  on generators, and  $\iota\sigma = \sigma\iota$ . Hence  $\rho^2 = \iota\sigma\iota\sigma = \sigma^2$ , and  $\sigma^2 = \text{Inn}(u)q^3\text{Inn}(u)q^3 = \text{Inn}(u)\text{Inn}(q^3u)q^6$ . Since  $q$  and  $\iota$  are automorphisms, so are  $\sigma$  and  $\rho$ .

In  $\text{Out } F(a, c)$  inner automorphisms vanish, so  $[\rho] = [\iota][q]^3$ . By Nielsen's theorem [17, 13] the abelianization map  $\text{Out } F(a, c) \rightarrow \text{GL}_2(\mathbb{Z})$  is an isomorphism; in the ordered basis  $(a, c)$  with columns the images of the generators,  $q$  abelianizes to  $Q$  and  $\iota$  to  $-I$ , so  $[\rho] = -Q^3$ .  $\square$

**Lemma 5.3** (The cocycle of the powers). *Put  $U_0 = 1$  and  $U_n = u q^3(u) q^6(u) \cdots q^{3(n-1)}(u)$ . Then for all  $n, n' \geq 0$*

$$\sigma^n = \text{Inn}(U_n) \circ q^{3n}, \quad U_{n+n'} = U_n q^{3n}(U_{n'}), \quad \rho^n = \iota^n \sigma^n.$$

*Proof.* Induction on  $n$  using  $q^3 \circ \text{Inn}(g) = \text{Inn}(q^3 g) \circ q^3$  gives the first two identities; the third is  $\iota\sigma = \sigma\iota$ .  $\square$

*Proof of Theorem C.* Theorem 5.2 gives the factorization, the class  $-Q^3$  and the formula for  $\rho^2$ ; the retraction  $r$  commutes with  $f$  by the computation of Section 5.1.  $\square$

**5.3. The cycle lattice.** The first homology of  $\Theta$  is the cycle lattice

$$H_1(\Theta; \mathbb{Z}) = \ker((1, -1, 1) : \mathbb{Z}^3 \rightarrow \mathbb{Z}) = \mathbb{Z}(A + B) \oplus \mathbb{Z}(B + C), \quad (5.1)$$

the classes of  $a$  and  $c$ . Write  $\text{ab}$  for the map induced on homology by an endomorphism of a free group, and use the ordered basis  $(a, c)$  with columns the images of the generators.

**Proposition 5.4** (Homology of the subdivision). *The matrix  $Q = q_{\text{ab}}$  has eigenvalues  $\phi$  and  $\psi$ , and the identification  $H_1(\Theta; \mathbb{Z}) = \mathcal{O}$ ,  $a \mapsto 1$ ,  $c \mapsto \phi$ , is the unique one with  $a \mapsto 1$  under which  $Q$  acts as multiplication by its expanding eigenvalue  $\phi$ . In it*

$$\rho_{\text{ab}} = -Q^3, \quad (\rho^n)_{\text{ab}} = (-Q)^{3n}, \quad (f_*)_{\text{ab}} = \text{diag}(-Q^3, -1) \text{ on } H_1(G; \mathbb{Z}) = \mathbb{Z}^3, \quad (5.2)$$

*the last matrix referring to the basis  $(a, c, e)$ ; thus one level of the hierarchy acts on the cycle lattice as multiplication by  $-\phi^3$ , and two levels as multiplication by  $\phi^6 = \theta^2$ .*

*Proof.* An additive map  $F : H_1(\Theta; \mathbb{Z}) \rightarrow \mathcal{O}$  with  $F(a) = 1$  and  $F(Qx) = \phi F(x)$  has  $F(c) = \phi$ , since  $Q(a) = c$ , hence  $F(ra + sc) = r + s\phi$  on the integral basis  $(a, c)$ ; conversely this formula is an additive isomorphism intertwining  $Q$  with multiplication by  $\phi$ , since  $\phi^2 = 1 + \phi$ . Replacing  $\phi$  by  $\psi$  gives the other identification with  $a \mapsto 1$ . The matrices follow from Theorem 5.2, Lemma 5.3 and Lemma 5.1, since  $\iota_{\text{ab}} = -I$  and inner automorphisms act trivially on homology.  $\square$

The sign in (5.2) is the reversal of the reference frame of Section 3, and  $\theta^2$  is the boundary growth of Theorem B over two levels.

## 6. THE TOWER OF CYCLIC COVERS

**6.1. Mapping tori and their cyclic covers.** Let  $T$  be the mapping torus of  $f|_{\Theta} : \Theta \rightarrow \Theta$  and  $Y$  the mapping torus of  $f : G \rightarrow G$ . Since  $f$  fixes  $v$ , the fundamental groups have the presentations

$$\pi_1(T) = \langle a, c, t \mid txt^{-1} = \rho(x) \rangle, \quad \pi_1(Y) = \langle a, c, e, t \mid txt^{-1} = f_*(x) \rangle, \quad (6.1)$$

the relations running over the generators, with the stable letter  $t$  oriented so that conjugation by  $t$  is the induced automorphism. For  $n \geq 1$  let  $T_n$  and  $Y_n$  be the  $n$ -fold cyclic covers of  $T$  and  $Y$  corresponding to the kernel of  $\pi_1 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/n$ , the first map sending  $t$  to 1 and the fibre to 0. Their fundamental groups are the subgroups generated by the fibre and  $t^n$ , with presentations as in (6.1) with  $\rho^n$  and  $f_*^n$  in place of  $\rho$  and  $f_*$  and  $t_n = t^n$  as stable letter. For  $m \mid n$  the covering  $T_n \rightarrow T_m$  induces the inclusion fixing the fibre and sending  $t_n$  to  $t_m^{n/m}$ . The retraction  $r$  and the inclusion  $\Theta \hookrightarrow G$  commute with  $f$  and induce retractions  $Y_n \rightarrow T_n$  with sections, compatible with all covering maps.

**Lemma 6.1.** *The spaces  $T_n$  and  $Y_n$  are aspherical.*

*Proof.* The infinite cyclic cover of the mapping torus of a graph map is its mapping telescope, an increasing union of subcomplexes homotopy equivalent to graphs; a map from a sphere of dimension at least two has compact image, hence lies in one of them and is null-homotopic. The higher homotopy groups of  $T_n$  and  $Y_n$  are those of the cover.  $\square$

**6.2. Realization on the once-punctured torus.** Let  $S = (\mathbb{R}^2/\mathbb{Z}^2) \setminus \{0\}$ , with base point  $p$  and  $\pi_1(S, p) = F(a, c)$  for a pair of standard curves. For  $A \in \mathrm{GL}_2(\mathbb{Z})$  let  $h_A: S \rightarrow S$  be the restriction of  $[x] \mapsto [Ax]$ , whose action on  $H_1(S; \mathbb{Z}) = \mathbb{Z}^2$  is  $A$ , and let  $M(A)$  be its mapping torus;  $h_A h_B = h_{AB}$ , so the  $k$ -fold cyclic cover of  $M(A)$  unwrapping the base circle is  $M(A^k)$ . It is a 3-manifold with one end, whose cross-section is a torus when  $\det A = 1$  and a Klein bottle when  $\det A = -1$ , and it is orientable exactly when  $\det A = 1$ . Two classical instances are the Gieseking manifold  $M(Q)$  and the complement of the figure-eight knot  $M(Q^2)$  [14, 22, 8]. The action of  $h_A$  on  $\pi_1(S, p)$  is  $[\gamma] \mapsto [\beta h_A(\gamma) \beta^{-1}]$  for a path  $\beta$  from  $p$  to  $h_A(p)$ , with  $\beta h_A(\beta) \cdots h_A^{n-1}(\beta)$  for  $h_A^n$ , so that  $(h_A^n)_* = ((h_A)_*)^n$ ; another path changes the action by an inner automorphism.

**Proposition 6.2.** *For every automorphism  $\alpha$  of  $F(a, c)$  there is an element  $g \in F(a, c)$  with  $\alpha = \mathrm{Inn}(g) \circ h_*$  for  $h = h_{\alpha_{ab}}$ . The homeomorphism  $h$  preserves orientation exactly when  $\det \alpha_{ab} = 1$ .*

*Proof.* By Nielsen's theorem [17, 13] the class of  $\alpha$  in  $\mathrm{Out} F(a, c)$  is determined by  $\alpha_{ab} \in \mathrm{GL}_2(\mathbb{Z})$ . Take  $h = h_{\alpha_{ab}}$ ; its homology matrix is  $\alpha_{ab}$ , so  $h_*$  has the same class as  $\alpha$ , and its derivative is  $\alpha_{ab}$ , so it preserves orientation exactly when  $\det \alpha_{ab} = 1$ .  $\square$

### 6.3. Bundles and the figure-eight knot.

**Theorem 6.3** (Restatement of Theorem D, first part). *There are homotopy equivalences  $T_n \simeq M((-Q)^{3n})$ ,  $n \geq 1$ , compatible with the covering maps  $T_n \rightarrow T_m$  for  $m \mid n$  and, up to homotopy, with the maps to the circle. The bundle  $M((-Q)^{3n})$  is homeomorphic to  $M(Q^{3n})$ , the  $3n$ -fold cyclic cover of the Gieseking manifold; it is orientable exactly when  $n$  is even, and for  $n = 2m$  it is the  $3m$ -fold cyclic cover of the figure-eight knot complement.*

*Proof.* Take  $h = h_{-Q^3}$  and, by Proposition 6.2,  $g$  with  $\rho = \mathrm{Inn}(g) \circ h_*$ . Induction gives  $\rho^n = \mathrm{Inn}(g_n) \circ h_*^n$  with  $g_n = g h_*(g) \cdots h_*^{n-1}(g)$ , and  $g_n = g_m h_*^m(g_m) \cdots h_*^{(k-1)m}(g_m)$  for  $n = km$ . The  $n$ -fold cover of  $M(-Q^3)$  is the mapping torus of  $h^n$ , with fundamental group  $\langle a, c, s_n \mid s_n x s_n^{-1} = h_*^n(x) \rangle$ . The assignment

$$\Psi_n(a) = a, \quad \Psi_n(c) = c, \quad \Psi_n(t_n) = g_n s_n \tag{6.2}$$

respects the relations, since  $g_n s_n x s_n^{-1} g_n^{-1} = g_n h_*^n(x) g_n^{-1} = \rho^n(x)$ , and has the inverse  $s_n \mapsto g_n^{-1} t_n$ ; so  $\Psi_n$  is an isomorphism. For  $n = km$  the covering inclusions send  $t_n \mapsto t_m^k$  and  $s_n \mapsto s_m^k$ , and  $\Psi_m(t_m)^k = (g_m s_m)^k = g_m h_*^m(g_m) \cdots h_*^{(k-1)m}(g_m) s_m^k = g_n s_m^k = \Psi_n(t_n)$ , so the isomorphisms commute with the inclusions. Both  $T$  and  $M(-Q^3)$  are aspherical CW complexes (Lemma 6.1; the bundle is covered by  $S \times \mathbb{R}$ , and  $S$  retracts onto a graph), so  $\Psi_1$  is induced by a based homotopy equivalence  $T \rightarrow M(-Q^3)$ ; it preserves the homomorphism to  $\mathbb{Z}$ , so the maps to the circle are homotopic,  $S^1$  being a  $K(\mathbb{Z}, 1)$ , and with compatible base points the equivalence lifts to the cyclic covers, the lifts inducing  $\Psi_n$  and commuting with the covering maps by the uniqueness of based lifts. Since  $h^n = h_{(-Q)^{3n}}$ , the mapping torus of  $h^n$  is  $M((-Q)^{3n})$ . Its orientation character is  $\det(-Q)^{3n} = (-1)^n$ . For  $n = 2m$ ,  $(-Q)^{6m} = (Q^2)^{3m} = Q^{6m}$ , and the covers are as stated. For odd  $n$ ,  $(-Q)^{3n} = -Q^{3n}$  and  $J(-Q^{3n})J^{-1} = Q^{-3n}$  with  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , since  $JQJ^{-1} = -Q^{-1}$ ; conjugating the monodromy by  $h_J$  and reversing the base circle gives homeomorphisms  $M(-Q^{3n}) \cong M(Q^{-3n}) \cong M(Q^{3n})$ .  $\square$

**Corollary 6.4** (Orientation and the norm). *Let  $N$  be the norm of  $\mathbb{Q}(\sqrt{5})$  over  $\mathbb{Q}$ . The orientation character of  $M((-Q)^{3n})$  is  $N((- \phi)^{3n}) = (-1)^n$ , the norm of the unit by which the monodromy of  $T_n$  acts on the cycle lattice.*

*Proof.* The determinant of the matrix of multiplication by an element of  $\mathcal{O}$  is its norm, and  $(-Q)^{3n}$  is the matrix of multiplication by  $(-\phi)^{3n}$  by Proposition 5.4.  $\square$

*Remark 6.5.* For  $n = 2m$ , filling the cusp of the bundle along the lift of the meridian gives the  $3m$ -fold cyclic branched cover  $\Sigma_{3m}(4_1)$  of the three-sphere along the figure-eight knot, whose fundamental group is Conway's Fibonacci group  $F(2, 6m)$  [10]; for  $m = 1$  it is the Hantzsche–Wendt manifold, with first homology  $(\mathbb{Z}/4)^2 = \mathcal{O}/(\phi^6 - 1)$ .

**6.4. Torsion homology.** For a mapping torus of a self-map  $h$  of a path-connected CW complex  $X$  the Wang sequence [9, Section 2.2] gives

$$H_1 \cong \mathbb{Z} \oplus H_1(X; \mathbb{Z}) / (h_* - 1)H_1(X; \mathbb{Z}), \quad (6.3)$$

the free summand being generated by the stable letter; when the second summand is finite it is the torsion subgroup and needs no choice of splitting.

**Theorem 6.6** (Restatement of Theorem D, second part). *With  $\mathcal{O} = \mathbb{Z}[\phi]$  acting on  $H_1(\Theta; \mathbb{Z})$  through Proposition 5.4,*

$$\mathrm{Tor} H_1(T_n; \mathbb{Z}) \cong \mathcal{O} / ((-\phi)^{3n} - 1) = \begin{cases} \mathcal{O} / (\phi^{3n} - 1), & n \text{ even}, \\ \mathcal{O} / (\psi^{3n} - 1), & n \text{ odd}, \end{cases} \quad \psi = -\phi^{-1}, \quad (6.4)$$

as  $\mathcal{O}$ -modules, and  $H_1(T_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Tor} H_1(T_n; \mathbb{Z})$ . For  $m \mid n$ , with  $s = \sum_{j < n/m} (-\phi)^{3mj}$ , the covering projection and the transfer are

$$[z]_n \mapsto [z]_m, \quad [z]_m \mapsto [sz]_n$$

in the notation of (6.4); projection is surjective, transfer is injective, and  $\pi_* \pi^! = (n/m) \mathrm{id}$ ,  $\pi^! \pi_* = \sum_{j < n/m} d^j$  with  $d$  the deck transformation acting by  $(-\phi)^{3m}$ . For the full endpoint graph,  $H_1(Y_n; \mathbb{Z}) \cong H_1(T_n; \mathbb{Z}) \oplus \mathbb{Z}$  for even  $n$  and  $H_1(T_n; \mathbb{Z}) \oplus \mathbb{Z}/2$  for odd  $n$ , and the retraction  $Y_n \rightarrow T_n$  induces the identity on  $\mathrm{Tor} H_1(T_n; \mathbb{Z})$ .

*Proof.* Apply (6.3) to  $T_n$  with  $X = \Theta$  and  $h_* = (\rho^n)_{\mathrm{ab}} = (-Q)^{3n}$ , using Proposition 5.4 to read  $(-Q)^{3n} - I$  as multiplication by  $(-\phi)^{3n} - 1$  on  $\mathcal{O}$ ; the quotient is finite because  $(-\phi)^{3n} \neq 1$ . The identifications  $\Psi_n$  of Theorem 6.3 are the identity on the fibre, so the same module is the torsion of  $H_1(M((-\phi)^{3n}); \mathbb{Z})$ , with the  $\mathcal{O}$ -action induced by  $Q$  on the fibre lattice; a generator of the deck group of  $T_n \rightarrow T$  acts on it by  $-\phi^3$ . For odd  $n$ ,  $\phi^{-3n} = (-\psi)^{3n} = -\psi^{3n}$ , so  $\phi^{3n} + 1 = \phi^{3n}(1 - \psi^{3n})$  and  $(-\phi)^{3n} - 1$  generates the ideal  $(\psi^{3n} - 1)$ . For  $n = km$  the identity  $((-\phi)^{3m} - 1)s = (-\phi)^{3n} - 1$  shows that reduction is well defined; a fibre cycle projects to itself, and its  $k$  lifts to the sheets of  $T_n \rightarrow T_m$  have classes  $(-\phi)^{3mj}z$ ,  $j < k$ , which gives the transfer, injective since  $s \neq 0$  in the domain  $\mathcal{O}$ , and the two composition formulas. For  $Y_n$  use (5.2): the summand of  $e$  contributes  $\mathrm{coker}((-1)^n - 1)$ , which is  $\mathbb{Z}$  for even  $n$  and  $\mathbb{Z}/2$  for odd  $n$ , and  $r_{n*}$  kills it and fixes the fibre classes of  $\Theta$ .  $\square$

*Proof of Theorem D.* Theorem 6.3 gives the equivalences, their coherence, the orientation and the covers of the figure-eight knot complement; Theorem 6.6 gives the torsion modules, the projections and the transfers.  $\square$

*Remark 6.7* (The two roots). The identification of the cycle lattice with  $\mathcal{O}$  in Proposition 5.4 is one of two with  $a \mapsto 1$ ; the other,  $c \mapsto \psi$ , makes the monodromy of  $T_n$  act by  $(-\psi)^{3n} = \phi^{-3n}$  and the annihilator equal to  $(\phi^{3n} - 1)$  at every level, and the two are exchanged by the Galois involution of  $\mathcal{O}$ , the restriction of the automorphism  $\tau$  of Section 4. If  $\varepsilon_n$  is the unit by which the monodromy of  $T_n$  acts, then  $N(\varepsilon_n) = (-1)^n$  (Corollary 6.4) and  $\tau(\varepsilon_n) = N(\varepsilon_n)\varepsilon_n^{-1}$ , so the annihilator  $(\varepsilon_n - 1)$  is Galois-stable at every orientable level, whereas at a non-orientable level its conjugate is  $(\varepsilon_n + 1)$ , a different ideal whenever a split prime divides  $L_{3n}$ , as 19 does for  $n = 3$  (Example 7.6).

## 7. THE ARITHMETIC OF THE TOWER

Throughout,  $\mathcal{O} = \mathbb{Z}[\phi]$ ,  $\psi = -\phi^{-1}$  is the conjugate root of  $x^2 = x + 1$ ,  $\mathcal{O}_j = \mathcal{O}/(\phi^j - 1)$  for  $j \geq 1$ , and  $F_j, L_j$  are the Fibonacci and Lucas numbers with  $F_0 = 0, F_1 = 1, L_0 = 2, L_1 = 1$ ; ring statements about torsion homology refer to the quotient-ring structure transported by the identification of Proposition 5.4. In the basis  $(1, \phi)$  multiplication by  $\phi$  is  $Q$ , and

$$Q^j = \begin{pmatrix} F_{j-1} & F_j \\ F_j & F_{j+1} \end{pmatrix}, \quad \text{tr } Q^j = L_j, \quad \det Q^j = (-1)^j.$$

**Lemma 7.1** (Galois twist). *The Galois involution  $\tau : \phi \mapsto \psi$  of  $\mathcal{O}$  induces ring isomorphisms  $\mathcal{O}/(\psi^j - 1) \cong \mathcal{O}_j$  for all  $j$ . Hence for every  $n$  the torsion homology of  $T_n$  is isomorphic as a ring to  $\mathcal{O}_{3n}$ , and as an  $\mathcal{O}$ -module to  $\mathcal{O}_{3n}$  when  $n$  is even and to  $\tau^*\mathcal{O}_{3n}$ , the same ring with the  $\mathcal{O}$ -action twisted by  $\tau$ , when  $n$  is odd.*

*Proof.*  $\tau$  carries the ideal  $(\psi^j - 1)$  onto  $(\phi^j - 1)$ ; the rest is Theorem 6.6.  $\square$

7.1. Four descriptions of  $\mathcal{O}_j$ .

**Proposition 7.2.** *For every  $j \geq 1$  there are isomorphisms*

- (a)  $\mathcal{O}_j \cong \mathbb{Z}^2/(Q^j - I)\mathbb{Z}^2$ ;
- (b)  $\mathcal{O}_j \cong \text{Fix}(Q^j : \mathbb{R}^2/\mathbb{Z}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2)$ , given by  $[z] \mapsto (Q^j - I)^{-1}z \bmod \mathbb{Z}^2$ ;
- (c)  $\mathcal{O}_j \cong F(2, j)^{\text{ab}}$ , the abelianization of the Fibonacci group  $F(2, j) = \langle x_1, \dots, x_j \mid x_i x_{i+1} = x_{i+2} \ (i \bmod j) \rangle$  of [6];
- (d) for even  $j$ ,  $\mathcal{O}_j \cong \text{Tor } H_1(N_{j/2}; \mathbb{Z})$ , where  $N_k$  is the  $k$ -fold cyclic cover of the figure-eight knot complement.

*The first three commute with multiplication by  $\phi$ , acting on the right sides through  $Q$ , through  $Q$  on the torus, and through the cyclic shift  $x_i \mapsto x_{i+1}$ .*

*Proof.* (a) In the basis  $(1, \phi)$  the matrix  $Q^j - I$  is multiplication by  $\phi^j - 1$ . (b) The fixed points of  $Q^j$  on the torus are the classes of the  $x \in \mathbb{R}^2$  with  $(Q^j - I)x \in \mathbb{Z}^2$ , that is  $(Q^j - I)^{-1}\mathbb{Z}^2/\mathbb{Z}^2$ , and  $z \mapsto (Q^j - I)^{-1}z$  induces the isomorphism from (a); it commutes with  $Q$ . (c) Abelianizing  $F(2, j)$  gives the quotient of  $\mathbb{Z}^j$  by the relations  $x_i + x_{i+1} - x_{i+2}$ . Writing  $x_i = t^{i-1}$  identifies  $\mathbb{Z}^j$  with  $\mathbb{Z}[t]/(t^j - 1)$  and the relations with the ideal generated by  $1 + t - t^2$ , so  $F(2, j)^{\text{ab}} \cong \mathbb{Z}[t]/(t^j - 1, t^2 - t - 1) = \mathcal{O}_j$  with  $t \mapsto \phi$ ; the shift is multiplication by  $t$ . (d)  $N_k = M(Q^{2k})$  by the description of the cyclic covers in Theorem 6.3, and (6.3) with (a) gives the torsion.  $\square$

## 7.2. Orders and invariant factors.

**Proposition 7.3.** *For every  $j \geq 1$ ,  $|\mathcal{O}_j| = |L_j - 1 - (-1)^j|$ , and as an abelian group*

$$\mathcal{O}_j \cong \begin{cases} \mathbb{Z}/L_j, & j \text{ odd}, 3 \nmid j, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/(L_j/2), & j \text{ odd}, 3 \mid j, \\ (\mathbb{Z}/L_{j/2})^2, & j \equiv 2 \pmod{4}, \\ \mathbb{Z}/F_{j/2} \oplus \mathbb{Z}/5F_{j/2}, & j \equiv 0 \pmod{4}, \end{cases}$$

*with  $\mathbb{Z}/1 = 0$ . Consequently the torsion homology of  $T_n$  has order  $L_{3n}$  for odd  $n$  and  $L_{3n} - 2$  for even  $n$ , and invariant factors  $(2, L_{3n}/2)$  for odd  $n$ ,  $(L_{3m}, L_{3m})$  for  $n = 2m$  with  $m$  odd, and  $(F_{3m}, 5F_{3m})$  for  $n = 2m$  with  $m$  even.*

*Proof.*  $\det(Q^j - I) = 1 - \text{tr } Q^j + \det Q^j = 1 - L_j + (-1)^j$ . For odd  $j$  the first invariant factor  $\delta_1$  is the greatest common divisor of the entries of  $Q^j - I$ ; the congruence  $Q^j \equiv I \pmod{\delta_1}$  forces  $(-1)^j = \det Q^j \equiv 1$ , so  $\delta_1 \mid 2$ , and  $Q$  has order exactly three modulo 2, so  $\delta_1 = 2$  exactly when  $3 \mid j$ . For  $j = 2m$ ,  $\phi^{2m} - 1 = \phi^m(\phi^m - \phi^{-m})$ , and  $\phi^{-m} = (-\psi)^m$  gives  $\phi^m - \phi^{-m} = L_m$  for odd

$m$  and  $\sqrt{5}F_m$  for even  $m$ ;  $\phi^m$  is a unit, and multiplication by  $\sqrt{5} = 2\phi - 1$  has matrix  $\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$  with invariant factors  $(1, 5)$ . The statements for  $T_n$  follow with  $j = 3n$ , using the ring isomorphism  $\mathcal{O}/(\psi^{3n} - 1) \cong \mathcal{O}_{3n}$  for odd  $n$ , together with  $L_{6m} - 2 = L_{3m}^2$  for odd  $m$  and  $L_{6m} - 2 = 5F_{3m}^2$  for even  $m$ ; the first six levels are listed in Figure 3(e).  $\square$

**7.3. Prime occurrence.** The levels at which a prime divides the orders of Proposition 7.3 are governed by the rank of apparition of the Lucas sequences [12, 5, 23].

**Proposition 7.4.** *Let  $p \neq 2, 5$  be prime and  $\varepsilon_p = \left(\frac{5}{p}\right)$ .*

- (a) *Let  $d_p$  be the order of  $\phi^6$  in  $(\mathcal{O}/p\mathcal{O})^\times$ . Then  $d_p \mid p - \varepsilon_p$ , and for even  $n = 2m$  the group  $\text{Tor } H_1(T_n; \mathbb{Z})$  has  $p$ -torsion exactly when  $d_p \mid m$ , in which case its reduction modulo  $p$  has dimension two over  $\mathbb{F}_p$ .*
- (b) *For each prime ideal  $\mathfrak{p}$  of  $\mathcal{O}$  above  $p$  let  $d_{\mathfrak{p}}$  be the order of  $-\phi^3$  in  $(\mathcal{O}/\mathfrak{p})^\times$ . The orders  $d_{\mathfrak{p}}$  for the one or two primes above  $p$  divide one another, and  $\text{Tor } H_1(T_n; \mathbb{Z})$  has  $p$ -torsion exactly when  $n$  is a multiple of the least of them. When it occurs, its reduction modulo  $p$  has dimension two over  $\mathbb{F}_p$  when  $p$  is inert or when  $n$  is a multiple of both orders, and dimension one otherwise.*

*Proof.* The polynomial  $x^2 - x - 1$  is separable modulo  $p$ , so  $\mathcal{O}/p\mathcal{O}$  is  $\mathbb{F}_p \times \mathbb{F}_p$  if  $\varepsilon_p = 1$  and  $\mathbb{F}_{p^2}$  if  $\varepsilon_p = -1$ . (a) The unit  $\phi^6$  has norm one, so its two coordinates in the split case are  $(\varrho, \varrho^{-1})$ , of the same order dividing  $p - 1$ ; in the inert case conjugation is the Frobenius,  $\phi^{6(p+1)} = N(\phi^6) = 1$ , and the order divides  $p + 1$ . In both cases  $\phi^{6m} - 1$  is either zero or invertible in  $\mathcal{O}/p\mathcal{O}$ , the first exactly when  $d_p \mid m$ , and reducing  $\mathcal{O}_{6m}$  modulo  $p$  gives the claim. (b) Now  $-\phi^3$  has norm  $-1$ , so in the split case its coordinates are  $(\varsigma, -\varsigma^{-1})$  for some  $\varsigma \in \mathbb{F}_p^\times$ , and the orders of  $\varsigma$  and of  $-\varsigma^{-1}$  are those of  $\varsigma$  and of  $-\varsigma$ , which differ by a factor of 1, 2 or  $\frac{1}{2}$ ; in the inert case there is one prime and one order. The ring  $\mathcal{O}/(p, (-\phi)^{3n} - 1)$  is nonzero exactly when  $(-\phi)^{3n} - 1$  lies in some  $\mathfrak{p}$  above  $p$ , that is when some  $d_{\mathfrak{p}}$  divides  $n$ , and its dimension over  $\mathbb{F}_p$  is the number of primes  $\mathfrak{p}$  above  $p$  containing  $(-\phi)^{3n} - 1$ , counted with the residue degree.  $\square$

*Example 7.5.* The prime 11 splits,  $\phi \equiv 4, 8 \pmod{11}$ , and  $-\phi^3 \equiv 2, 5$ , of orders 10 and 5. So 11-torsion first appears at level 5, where  $L_{15} = 1364 = 2^2 \cdot 11 \cdot 31$ , with  $\text{Tor } H_1(T_5; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/682$  and a cyclic 11-part; on the orientable levels it first appears at level 10, where it has dimension two. The order of  $-\phi^3$  in  $(\mathcal{O}/11\mathcal{O})^\times$  is 10, so the least order of (b) cannot be replaced by the order in the full residue ring.

*Example 7.6.* The prime 19 splits,  $\phi \equiv 5, 15 \pmod{19}$ , and  $\phi^9 \equiv 1, -1$  respectively, so 19 divides  $L_9 = 76$ ; at level  $n = 3$  the ideal  $(\psi^9 - 1) = (\phi^9 + 1)$  is  $(2)\mathfrak{p}'$  while its conjugate  $(\phi^9 - 1)$  is  $(2)\mathfrak{p}$ , for the two primes  $\mathfrak{p}, \mathfrak{p}'$  above 19: the two identifications of Remark 6.7 assign different ideals to this non-orientable level.

#### 7.4. The integral lattice and its residue field.

**Proposition 7.7** (The integral monodromy order). *Put  $M = -Q^3$ ,  $\mu = -\phi^3$  and  $S = \mathbb{Z}[\mu] \subset \mathcal{O}$ . The integral commutant of  $M$  is*

$$\{X \in M_2(\mathbb{Z}) : XM = MX\} = \mathbb{Z}[Q] \cong \mathcal{O}, \quad S = \mathbb{Z} + 2\mathbb{Z}\phi, \quad [\mathcal{O} : S] = 2. \quad (7.1)$$

*Multiplication by  $\mu$  on  $S$  and on  $\mathcal{O}$  has the characteristic polynomial  $t^2 + 4t - 1$  on both, but  $S/(\mu - 1)S \cong \mathbb{Z}/4$  while  $\mathcal{O}/(\mu - 1)\mathcal{O} \cong \mathbb{F}_4$ , and for every  $n \geq 1$  the torsion ring  $R_n = \mathcal{O}/(\mu^n - 1)$  of  $T_n$  satisfies  $R_n/2R_n \cong \mathcal{O}/2\mathcal{O} \cong \mathbb{F}_4$ .*

*Proof.* Since  $Q^2 = Q + I$ ,  $M = -I - 2Q$ , and  $M_2(\mathbb{Z})$  has no additive 2-torsion, so a matrix commutes with  $M$  exactly when it commutes with  $Q$ ; direct multiplication shows that the commutant of  $Q$  consists of the matrices  $\begin{pmatrix} a & b \\ b & a+b \end{pmatrix}$ , which is  $\mathbb{Z}[Q]$ , identified with  $\mathcal{O}$  by  $Q \mapsto \phi$  as in Proposition 5.4.

Next  $\mu = -1 - 2\phi$  and  $\mu^2 + 4\mu - 1 = 0$ , so  $S = \mathbb{Z} + \mathbb{Z}\mu = \mathbb{Z} + 2\mathbb{Z}\phi$  has index two, and both lattices span  $\mathbb{Q}(\sqrt{5})$ , on which  $\mu$  has characteristic polynomial  $t^2 + 4t - 1$ . The presentation  $S = \mathbb{Z}[t]/(t^2 + 4t - 1)$  gives  $S/(\mu - 1)S = \mathbb{Z}/4$ , whereas in  $\mathcal{O}$ ,  $\mu - 1 = -2\phi^2$  and  $\phi$  is a unit, so  $(\mu - 1)\mathcal{O} = 2\mathcal{O}$ , and  $\mathcal{O}/2\mathcal{O} \cong \mathbb{F}_4$  because  $t^2 - t - 1$  is irreducible modulo 2. Finally  $\mu \equiv 1 \pmod{2\mathcal{O}}$ , so  $(2, \mu^n - 1) = 2\mathcal{O}$  for every  $n$ .  $\square$

### 7.5. When the torsion is a field.

**Corollary 7.8.**  $\mathcal{O}_j$  is a field exactly when  $j \in \{3, 4\}$  or  $j = p \geq 5$  is a prime with  $L_p$  prime; in the last case  $\mathcal{O}_p \cong \mathbb{F}_{L_p}$ . Exactly one torsion ring of the tower is a field,  $\text{Tor } H_1(T_1; \mathbb{Z}) = \mathcal{O}/2\mathcal{O} \cong \mathbb{F}_4$ .

*Proof.* The additive group of a finite field is elementary abelian, and a unital ring whose additive group is cyclic of order  $N$  is  $\mathbb{Z}/N$ : if  $g$  generates the group,  $1 = cg$  and  $g^2 = dg$ , then  $g = cdg$ , so  $c$  is invertible modulo  $N$  and 1 generates. The rings  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are zero, since  $\phi - 1 = \phi^{-1}$  and  $\phi^2 - 1 = \phi$  are units, and are not fields; let  $j \geq 3$ . In the four cases of Proposition 7.3: for odd  $j$  with  $3 \nmid j$ ,  $\mathcal{O}_j \cong \mathbb{Z}/L_j$  as a ring, a field exactly when  $L_j$  is prime, and then  $j$  is prime, since  $L_m \mid L_j$  whenever  $j/m$  is odd and  $L_m > 1$  for  $m > 1$ ; for odd  $j$  with  $3 \mid j$ ,  $\mathbb{Z}/2 \oplus \mathbb{Z}/(L_j/2)$  is elementary abelian only for  $L_j = 4$ , that is  $j = 3$ , and  $\mathcal{O}_3 = \mathcal{O}/(2\phi) = \mathcal{O}/2\mathcal{O} = \mathbb{F}_4$  since 2 is inert; for  $j \equiv 2 \pmod{4}$ ,  $j = 2m$ , the group  $(\mathbb{Z}/L_m)^2$  is elementary abelian only for  $L_m$  prime, but then  $L_m^2 - 5F_m^2 = -4$  shows that 5 is a square modulo  $L_m$ , so  $L_m$  splits in  $\mathcal{O}$  and  $\mathcal{O}_{2m} = \mathcal{O}/L_m\mathcal{O} \cong \mathbb{F}_{L_m} \times \mathbb{F}_{L_m}$  is not a field; for  $j \equiv 0 \pmod{4}$ ,  $\mathbb{Z}/F_{j/2} \oplus \mathbb{Z}/5F_{j/2}$  is elementary abelian only for  $F_{j/2} = 1$ , that is  $j = 4$ , and  $\mathcal{O}_4 = \mathcal{O}/(\phi^2\sqrt{5}) = \mathcal{O}/(\sqrt{5}) \cong \mathbb{F}_5$ . The levels of the tower are the indices  $3n$ , of which only 3 is in the list.  $\square$

### 7.6. Dimension as torsion growth.

**Theorem 7.9** (Restatement of Theorem E(v)). *Let  $\delta$  be the Hausdorff dimension of the limiting boundary curve  $K$  of Theorem 4.6, let  $R(\mathbb{Q}(\sqrt{5})) = \log \phi$  and  $R(\mathbb{Q}(\sqrt{15})) = \log(4 + \sqrt{15})$  be the regulators, and let  $M$  denote the Mahler measure. Then*

$$\delta = \lim_{n \rightarrow \infty} \frac{\log |\text{Tor } H_1(T_n; \mathbb{Z})|}{n \log \sqrt{\lambda}} = \lim_{n \rightarrow \infty} \frac{\log L_{3n}}{n \log \sqrt{\lambda}} = \frac{6 R(\mathbb{Q}(\sqrt{5}))}{R(\mathbb{Q}(\sqrt{15}))} = \frac{3 \log M(t^2 - 3t + 1)}{\log \lambda}.$$

*Proof.* By Proposition 7.3,  $\log |\text{Tor } H_1(T_n; \mathbb{Z})| = \log L_{3n} + O(1) = 3n \log \phi + O(1)$ , so the first limit is  $6 \log \phi / \log \lambda$ , which is  $\delta$  by Theorem 4.6. The fundamental units of  $\mathbb{Q}(\sqrt{5})$  and  $\mathbb{Q}(\sqrt{15})$  are  $\phi$  and  $4 + \sqrt{15}$ , the latter because  $(4, 1)$  is the least solution of  $x^2 - 15y^2 = 1$  and  $x^2 - 15y^2 = -1$  has no solution modulo 3. Finally  $t^2 - 3t + 1 = (t - \phi^2)(t - \phi^{-2})$  has Mahler measure  $\phi^2$ .  $\square$

Per degree of the cover of the figure-eight knot complement the torsion grows at the rate

$$\lim_{m \rightarrow \infty} \frac{\log |\text{Tor } H_1(T_{2m}; \mathbb{Z})|}{3m} = \log M(t^2 - 3t + 1),$$

the logarithmic Mahler measure of the Alexander polynomial, as for cyclic covers of knots in general [16, 19, 18]; here the orders are known exactly, and normalizing by the expansion of the tile instead gives  $\delta$ . The dimension  $\delta$  is also the entropy  $3 \log \phi$  of the toral automorphism  $Q^3$  divided by the expansion exponent  $\log \sqrt{\lambda}$  of one level, as in Bowen's formula for quasi-circles [4].

*Proof of Theorem E.* Parts (i)–(iv) are Propositions 7.2, 7.3, 7.4 and 7.7 and Corollary 7.8; part (v) is Theorem 7.9.  $\square$

## APPENDIX A. EXACT DATA FOR THE POLYGON

With  $s = \sqrt{3}$  the vertices (2.1) of  $H$  are those of Table 3.

| $j$                     | 0 | 1       | 2         | 3         | 4         | 5         | 6         |
|-------------------------|---|---------|-----------|-----------|-----------|-----------|-----------|
| $\operatorname{Re} b_j$ | 0 | 1       | $3/2$     | $(3+s)/2$ | $(3+s)/2$ | $(5+s)/2$ | $(6+s)/2$ |
| $\operatorname{Im} b_j$ | 0 | 0       | $-s/2$    | $(1-s)/2$ | $(3-s)/2$ | $(3-s)/2$ | $3/2$     |
| $j$                     | 7 | 8       | 9         | 10        | 11        | 12        | 13        |
| $\operatorname{Re} b_j$ | 3 | $3-s/2$ | $(5-s)/2$ | $(3-s)/2$ | $(1-s)/2$ | $-s/2$    | 0         |
| $\operatorname{Im} b_j$ | 2 | $3/2$   | $(3+s)/2$ | $(3+s)/2$ | $(3+s)/2$ | $3/2$     | 1         |

TABLE 3. The vertices  $b_0, \dots, b_{13}$  of  $H$ , with  $s = \sqrt{3}$ .

At  $c = c_0$  form  $q_j$  by (2.4) and let  $(v_0, \dots, v_{26})$  be the alternating list of the  $b_j$  and  $q_j$  with  $b_{10}$  deleted; all coordinates lie in  $\mathbb{Q}(\sqrt{3})$ . For the side  $[v_a, v_{a+1}]$ , indices modulo 27, and a coordinate  $\star \in \{x, y\}$  let  $I_{a,\star}$  be the closed interval spanned by the  $\star$ -coordinates of its endpoints, and for intervals put  $\Delta([u, v], [r, t]) = \max(r - v, u - t)$ , the signed gap.

**Proposition A.1.** *Over the 324 pairs  $0 \leq a < b < 27$  of nonadjacent sides, the quantity  $g_{ab} = \max\{\Delta(I_{a,x}, I_{b,x}), \Delta(I_{a,y}, I_{b,y})\}$  satisfies*

$$\min_{a,b} g_{ab} = \frac{3 - \sqrt{3}}{8},$$

*attained at  $(a, b) \in \{(8, 10), (19, 21), (19, 22), (24, 26)\}$ . The shoelace sum  $\frac{1}{2} \sum_a (v_{a,x} v_{a+1,y} - v_{a,y} v_{a+1,x})$  equals  $3 + 3\sqrt{3}$ ; the squared lengths of the maximal sides are  $1/8$ ,  $5/8$  and  $5/2$  with multiplicities 14, 12 and 1; and consecutive side vectors are never parallel.*

*Proof.* Every quantity involved is of the form  $a + b\sqrt{3}$  with  $a, b \in \mathbb{Q}$ , and its sign is  $\operatorname{sgn} b$  if  $a = 0$ ,  $\operatorname{sgn} a$  if  $b = 0$  or if  $a, b$  have the same sign, and  $\operatorname{sgn}(a) \operatorname{sgn}(a^2 - 3b^2)$  otherwise,  $a^2 = 3b^2$  being impossible for nonzero rationals. All minima, maxima and comparisons in the statement therefore reduce to a finite amount of rational arithmetic, which can be verified with the accompanying scripts.  $\square$

Since every pair of nonadjacent sides has disjoint coordinate projections in at least one direction, no two of them meet; the bound (2.8) extends this from  $c_0$  to all of  $U$ , as in the proof of Proposition 2.1.

## APPENDIX B. THE BOUNDARY AT EVERY LEVEL

This appendix contains the exact computations in  $\mathcal{F}$  behind Lemma 4.1, Proposition 4.2 and Lemma 4.3, and the induction that carries the subdivision from level one to every level. The computations can be verified with the accompanying scripts.

**B.1. Exact computations in  $\mathcal{F}$ .** In the  $\mathbb{Q}$ -basis  $1, \sqrt{3}, \sqrt{5}, \sqrt{15}, i, i\sqrt{3}, i\sqrt{5}, i\sqrt{15}$  of  $\mathcal{F}$ , indexed by  $a \in \{0, \dots, 7\}$  with binary digits  $a_0, a_1, a_2$ ,

$$e_a e_b = 3^{a_0 b_0} 5^{a_1 b_1} (-1)^{a_2 b_2} e_{a \operatorname{xor} b}, \quad (\text{B.1})$$

so that multiplication, complex conjugation (negating the coefficients with  $a_2 = 1$ ),  $\tau$  (negating those with  $a_1 = 1$ ) and inversion (multiplying by the conjugates in  $i, \sqrt{5}$  and  $\sqrt{3}$ ) are rational arithmetic on eight coefficients. The sign of a nonzero real element  $q_0 + q_1\sqrt{3} + q_2\sqrt{5} + q_3\sqrt{15}$  is decided by enclosing the three radicals in rational intervals of width  $2^{-b}$  and doubling  $b$  until the interval sum excludes zero, which terminates since  $1, \sqrt{3}, \sqrt{5}, \sqrt{15}$  are linearly independent over  $\mathbb{Q}$ .

**Lemma B.1** (Identities). *The following hold in  $\mathcal{F}$ :  $R1 = 1$ ;  $R^2d = 8Rd - d$  and the four rows of  $Rd$  in the proof of Lemma 4.1;  $-\overline{E\xi} = c_*\xi$  and  $c_*\overline{c_*} = \lambda$ ;  $g_jp_j = p_j$  for  $j \in \{2, 4, 7\}$ ; the twenty endpoint identities of the limit maps, twelve between consecutive entries of Table 2 and eight between extreme entries and parent key points; and  $\nu k_0 = 0$ ,  $\nu\overline{E} = (\zeta - 1)\overline{\nu}$  for  $\nu = (1, \zeta - 2, 1 - \zeta, 0)$ .*

*Proof.* Each is an equality of elements or vectors of  $\mathcal{F}$ , computed from the definitions by (B.1) and compared coefficientwise.  $\square$

**Lemma B.2** (Convexity, nesting and separation). *Put  $[a, b, z] = \text{Im}(\overline{(b - a)}(z - a))$ .*

- (i) *For each list  $C_i = (z_0, \dots, z_{q_i-1})$  in (4.6),  $(q_i) = (7, 7, 8, 8)$ , and all  $j, k$  modulo  $q_i$ ,  $[z_j, z_{j+1}, z_k]$  is zero for  $k \in \{j, j+1\}$  and positive otherwise (226 determinants).*
- (ii) *For each entry  $(\sigma, j)$  of the word of  $i$ , every vertex of  $g_\sigma(C_j)$  satisfies the supporting inequalities  $[z_k, z_{k+1}, \cdot] \geq 0$  of  $C_i$  (118 vertices, 900 inequalities).*
- (iii) *The initial endpoint of the word of  $i$  lies in  $g_\sigma(C_j)$  only for its first entry and the terminal endpoint only for its last (32 membership tests).*
- (iv) *For each of the six pairs of root polygons and the thirty pairs of entries of one word, some edge  $z_k z_{k+1}$  of the first polygon has  $[z_k, z_{k+1}, \cdot]$  negative at every vertex of the second, or, for consecutive entries and cyclic root pairs, nonpositive with a single zero at the prescribed common endpoint.*

*Proof.* All points lie in  $\mathcal{F}$ ; every sign is decided by the procedure above.  $\square$

Lemma B.2 implies Lemma 4.3: by (i) each  $C_i$  is strictly convex, counterclockwise and the intersection of its supporting half-planes; (ii) and convexity give the nesting of  $g_\sigma(C_j)$ ; (iv) and convexity give the intersections, and the single zero is the prescribed common endpoint, which lies in both polygons by Lemma B.1.

**B.2. The subdivision at every level.** The level-zero singleton is the closed fourteen-edge path (2.1) with  $b_{10}$  retained; its arcs between  $b_3, b_5, b_7, b_{11}$  have 2, 2, 4, 6 unit edges. The level-zero Mystic has twenty exposed unit edges and the same six-edge  $D$ -arc.

**Proposition B.3** (Level one). *Under the placements  $f_{0,\sigma}$ , the seven Spectre children and the Mystic child of the Spectre cluster contain nine copies of  $H$  with 126 oriented unit edges; 40 oppositely directed pairs cancel, 46 edges remain, and the words of Table 2 expand to 8, 8, 18 and 12 edges whose multiset of directed edges is the exposed multiset. In the Mystic cluster, slot 2 omitted, eight copies contribute 112 edges, 34 pairs cancel, 44 edges remain, and the words with (4.5) as third word have lengths 8, 8, 16, 12 with the same multiset. In each cluster the exposed directed edges form a single cycle, every exposed vertex having one incoming and one outgoing edge, and the concatenated words follow this cycle from the first parent key through the remaining keys in order.*

*Proof.* Every key point lies in  $\mathbb{Q}(\omega)$ , so the directed unit edges are determined exactly by their endpoints, and the multiset equality is a finite comparison; following the unique outgoing exposed edge at each vertex gives the cycle and its passage through the parent keys.  $\square$

A direction word  $w = (m_1, \dots, m_q)$  with letters in  $\mathbb{Z}/12$  is the unit-step path with displacements  $\omega^{m_1}, \dots, \omega^{m_q}$  from a given initial point; words are concatenated by juxtaposition,  $\text{rev } w$  reverses the order of the letters, and  $\mathbf{r}_t(m) = m + t$ ,  $\mathbf{f}_t(m) = t - m$  act letterwise, so that reversed traversal has direction word  $\mathbf{r}_6 \text{rev } w$ . Let  $a_m, b_m, c_m, d_m$  be the direction words defined from

$$a_0 = (3, 0), \quad b_0 = (2, 5), \quad c_0 = (7, 4, 6, 6), \quad d_0 = (8, 11, 9, 0, 10, 1) \quad (\text{B.2})$$

by the slot rules of Table 2, a reversed child in slot  $\sigma$  contributing  $f_{-2a_\sigma} \text{rev } w$  because the linear part of its placement is  $z \mapsto -\overline{r_\sigma z}$ ; with a prime for the next level,

$$\begin{aligned} a' &= f_4 \text{rev } b \ f_6 \text{rev } c \ f_6 \text{rev } b, \\ b' &= f_6 \text{rev } a \ f_6 \text{rev } b \ f_8 \text{rev } c, \\ c' &= f_8 \text{rev } b \ f_{10} \text{rev } c \ f_{10} \text{rev } b \ f_{10} \text{rev } a \ f_{10} \text{rev } b \ f_0 \text{rev } c \ f_0 \text{rev } b. \end{aligned} \tag{B.3}$$

Proposition B.12 identifies these recursively defined words with the exposed boundaries.

**Lemma B.4** (A reversed pair and a symmetric remainder). *For every  $m \geq 0$  there is a direction word  $t_m$  with*

$$b_m = \mathbf{r}_2 \text{rev } a_m, \quad c_m = \mathbf{r}_4 a_m t_m, \quad \text{rev } t_m = t_m, \tag{B.4}$$

$t_0 = (6, 6)$ , and

$$\begin{aligned} a' &= f_2 a \ f_6 t \ f_2 \text{rev } a \ f_4 a, \\ t' &= f_{10} \text{rev } a \ f_8 a \ f_0 t \ f_8 \text{rev } a \ f_{10} a, \end{aligned} \tag{B.5}$$

as ordered words.

*Proof.* Induction on the level, starting from (B.2). Since  $\text{rev}$  commutes with letterwise maps,  $\text{rev } b = \mathbf{r}_2 a$  and  $\text{rev } c = t \mathbf{r}_4 \text{rev } a$ ; substitution in (B.3) gives (B.5) from the first line,  $b' = f_6 \text{rev } a \ f_4 a \ f_8 t \ f_4 \text{rev } a = \mathbf{r}_2 \text{rev } a'$  from the second, and from the third the decomposition  $c' = \mathbf{r}_4 a' t'$  with the displayed  $t'$ , whose outer and inner pairs of blocks are reversals of each other and whose middle block is a palindrome.  $\square$

**Lemma B.5** (No retracing). *The four singleton words and, at every positive level, the Mystic word are cyclically reduced in the direction alphabet with  $m^{-1} = m + 6$ .*

*Proof.* The first and last letters of the four words alternate between the two states of Table 4.

| $m \bmod 2$ | $a_m$   | $b_m$  | $c_m$   | $d_m$  |
|-------------|---------|--------|---------|--------|
| 0           | (3, 0)  | (2, 5) | (7, 6)  | (8, 1) |
| 1           | (11, 4) | (6, 1) | (3, 10) | (0, 9) |

TABLE 4. The first and last letters of the direction words at even and odd levels.

With the turn at a concatenation defined as first outgoing minus last incoming direction modulo twelve, the turns between successive children are 10, 2 in  $A$ ; 10, 10 in  $B$ ; 10, 2, 2, 10, 10, 2 in  $C$ ; 10, 10 in  $D$ ; 2, 2, 10, 2 in (4.5); and 2 at every join of successive root words, the cyclic one included. None is 6, the initial words have no adjacent inverse letters, and letterwise maps and reversal preserve this.  $\square$

**Corollary B.6** (Placed congruences). *With  $A_m, B_m, C_m$  placed at  $k_{m,0}, k_{m,1}, k_{m,2}$  and*

$$\beta_m(z) = -\zeta z + (1 + \zeta)k_{m,1}, \quad \beta'_m(z) = \zeta^2 z + k_{m,2} - \zeta^2 k_{m,0},$$

$B_m$  is  $\beta_m(A_m)$  traversed backwards,  $C_m$  begins with  $\beta'_m(A_m)$ , and the remainder of  $C_m$ , from  $\varrho_m = k_{m,2} + \zeta^2(k_{m,1} - k_{m,0})$  to  $k_{m,3}$ , is carried onto itself with reversed traversal by the half-turn about  $(\varrho_m + k_{m,3})/2$ .

*Proof.* Direction words agree by Lemma B.4 and initial points by construction;  $-\zeta k_{m,0} + (1 + \zeta)k_{m,1} = k_{m,2}$  is  $\nu k_m = 0$ ; the half-turn reverses the letter order of the remainder without changing the letters, which is  $\text{rev } t_m = t_m$ .  $\square$

**Lemma B.7** (The fourth arc). *Write  $w^{-1} = \mathbf{r}_6 \text{ rev } w$  and let  $\equiv$  be equality in the free group on the six unoriented directions, in which  $\mathbf{r}_j, \mathbf{f}_j$  act as automorphisms and  $\mathbf{r}_6 t = t^{-1}$ . For every  $m \geq 0$ ,*

$$d_m \equiv (\mathbf{r}_2 a_m)^{-1} \mathbf{r}_6 a_m t_m^{-1} (\mathbf{r}_4 a_m)^{-1}, \quad (\text{B.6})$$

and the closed six-arc word

$$d_m c_m \text{ rev } a_m \mathbf{r}_{10} c_m \mathbf{r}_2 \text{ rev } a_m \mathbf{r}_4 c_m \quad (\text{B.7})$$

reduces to the empty word.

*Proof.* At level zero the right side of (B.6) is  $(8, 11, 9, 6, 0, 0, 10, 1)$ , which reduces to  $d_0$ . Assume (B.6) at one level. The fourth slot rule and  $c = \mathbf{r}_4 a t$  give  $d' = \mathbf{f}_0 \text{ rev } a \mathbf{f}_8 \text{ rev } d \mathbf{f}_4 t \mathbf{f}_0 \text{ rev } a$ . With  $\mathbf{a}_j = \mathbf{f}_j a$ ,  $\mathbf{t}_j = \mathbf{f}_j t$ , indices modulo 12,  $\mathbf{t}_{j+6} = \mathbf{t}_j^{-1}$  and (B.5) reads  $a' = \mathbf{a}_2 \mathbf{t}_6 \mathbf{a}_8^{-1} \mathbf{a}_4$ ,  $t' = \mathbf{a}_4^{-1} \mathbf{a}_8 \mathbf{t}_0 \mathbf{a}_2^{-1} \mathbf{a}_{10}$ ; substituting the inductive expression for  $d$ ,

$$d' \equiv \mathbf{a}_6^{-1} \mathbf{a}_{10} \mathbf{t}_2 \mathbf{a}_8^{-1} \mathbf{a}_0 \mathbf{t}_4 \mathbf{a}_6^{-1},$$

and

$$\begin{aligned} (\mathbf{r}_2 a')^{-1} &= \mathbf{a}_6^{-1} \mathbf{a}_{10} \mathbf{t}_8^{-1} \mathbf{a}_4^{-1}, & \mathbf{r}_6 a' &= \mathbf{a}_8 \mathbf{t}_0 \mathbf{a}_2^{-1} \mathbf{a}_{10}, \\ (t')^{-1} &= \mathbf{a}_{10}^{-1} \mathbf{a}_2 \mathbf{t}_0^{-1} \mathbf{a}_8^{-1} \mathbf{a}_4, & (\mathbf{r}_4 a')^{-1} &= \mathbf{a}_8^{-1} \mathbf{a}_0 \mathbf{t}_{10}^{-1} \mathbf{a}_6^{-1}, \end{aligned}$$

whose product in this order reduces, with  $\mathbf{t}_8^{-1} = \mathbf{t}_2$  and  $\mathbf{t}_{10}^{-1} = \mathbf{t}_4$ , to the same expression. Solving (B.7) for  $d$  gives  $d \equiv \mathbf{r}_{10} \text{ rev } c \mathbf{r}_8 a \mathbf{r}_4 \text{ rev } c \mathbf{r}_6 a \mathbf{r}_6 \text{ rev } c$ ; substituting  $\text{rev } c = t \mathbf{r}_4 \text{ rev } a$  and cancelling  $(\mathbf{r}_2 \text{ rev } a)(\mathbf{r}_8 a)$  and  $(\mathbf{r}_{10} t)(\mathbf{r}_4 t)$  leaves  $\mathbf{r}_8 \text{ rev } a \mathbf{r}_6 a \mathbf{r}_6 t \mathbf{r}_{10} \text{ rev } a$ , the right side of (B.6).  $\square$

**B.3. Spherical diagrams for the two clusters.** A row  $(J, \pm, r, t)$  of Table 5 denotes the image of the placed path  $J_m \in \{A_m, B_m, C_m, D_m\}$  under  $z \mapsto rz + tk_m$ , traversed backwards when the sign is  $-$ ; the rows  $t$  are reduced modulo  $\nu$ , which annihilates every key vector. Let  $\mathcal{N}_m$  be the closed six-path word formed by the rows of Table 5 in order; the consecutive endpoint equalities are congruences of the rows modulo  $\nu$ .

| arc | sign | $r$         | $t_1$       | $t_2$       | $t_3$       |
|-----|------|-------------|-------------|-------------|-------------|
| $D$ | $+$  | 1           | 0           | 0           | 0           |
| $C$ | $+$  | 1           | $2 - \zeta$ | $\zeta - 2$ | 0           |
| $A$ | $-$  | $-1$        | $3 - \zeta$ | $\zeta - 2$ | 1           |
| $C$ | $+$  | $1 - \zeta$ | 1           | $\zeta - 2$ | 1           |
| $A$ | $-$  | $-\zeta$    | $1 + \zeta$ | $\zeta - 2$ | $2 - \zeta$ |
| $C$ | $+$  | $\zeta - 1$ | 0           | 0           | $2 - \zeta$ |

TABLE 5. The six placed paths of the relator  $\mathcal{N}_m$ .

**Lemma B.8** (An invariant relator). *For every  $m \geq 0$  the placed path  $\mathcal{N}_m$  reduces to a constant path by cancellation of adjacent inverse steps.*

*Proof.* With  $1 - \zeta = \omega^{10}$  and  $\zeta - 1 = \omega^4$  the direction word of Table 5 is (B.7), which reduces to the empty word by Lemma B.7; the endpoint equalities place each cancelling pair on one unit segment.  $\square$

Expanding the boundary of each child of the Spectre cluster by one level, with the Mystic word in slot 7, and the proposed parent boundary by the same amount, and applying Corollary B.6 with all occurrence orders retained, the difference between the sum of the child boundaries and the parent boundary is, in each of the two clusters, the sum of the six images of  $\mathcal{N}$  under the maps  $z \mapsto rz + tk$  of Table 6, an identity between formal sums of placed arcs valid at every level, since its terms are rows modulo  $\nu$ .

| $r$         | $t_1$        | $t_2$         | $t_3$         |
|-------------|--------------|---------------|---------------|
| $\zeta - 1$ | $3 - 3\zeta$ | $2\zeta - 1$  | 0             |
| $\zeta$     | $3 - \zeta$  | 0             | -2            |
| 1           | $2 + 2\zeta$ | $1 - 2\zeta$  | -3            |
| $\zeta$     | $2\zeta$     | $2 - \zeta$   | $-1 - 2\zeta$ |
| $-\zeta$    | $7 - 2\zeta$ | -3            | $3\zeta - 3$  |
| $1 - \zeta$ | 7            | $-2 - 2\zeta$ | $3\zeta - 5$  |

TABLE 6. The six placements of the relator  $\mathcal{N}$  in the difference of boundaries.

**Lemma B.9** (Spherical diagrams). *In each cluster the expanded child words, the reversed parent word and the six inverse relators are the faces of a spherical diagram in which every face has embedded boundary and the six closed relator faces are pairwise disjoint. In the Spectre diagram three relator faces meet the outer face in one vertex each and three do not meet it; in the Mystic diagram two meet it in one vertex each, three do not meet it, and one meets it in a path of two consecutive sides. At levels zero and one, folding the relator faces gives the rotation system of the marked cluster of Theorem 3.1, in both frames.*

*Proof.* Pairing occurrences of one placed arc with opposite traversal, every occurrence has exactly one mate, determined by Tables 2, 5 and 6 and by (4.5). The diagram has  $(V, E, F) = (102, 115, 15)$  for the Spectre cluster and  $(90, 102, 14)$  for the Mystic cluster, connected dual graph, circular vertex links and orientation-reversing identifications, so it is a sphere. Its outer face visits its 68, respectively 58, vertices once, each ordinary child face visits 16 distinct vertices, the Mystic child 14, and each relator six; the vertex sets of the six relators are pairwise disjoint, and their intersections with the outer vertex set have sizes 0, 0, 0, 1, 1, 1 in the Spectre diagram and 0, 0, 0, 1, 1, 3 in the Mystic diagram, where the three common vertices are joined by exactly two common sides, the consecutive  $C$ - and  $D$ -sides of the outer word, paired with the last and first sides of the relator of Table 6 with  $r = \zeta$  and  $t = (2\zeta, 2 - \zeta, -1 - 2\zeta)$ ; all of this is read off from the vertex classes generated by the pairing. Evaluating the paired arcs at levels zero and one, subdividing into unit steps and folding the relator faces, the remaining half-edge pairs are the oppositely directed unit segments of the cluster, its vertices are the vertices of the cluster without extra identification or splitting, and the rotation at every vertex is the clockwise order of the outgoing rays  $\omega^j$ ; these are finite comparisons in  $\mathbb{Q}(\omega)$ .  $\square$

**Lemma B.10** (Folding the relator faces). *Subdivide the spherical diagrams of Lemma B.9 into the unit steps of the paths at any level, and suppose that every placed child boundary is a simple closed curve and that the outer word is cyclically reduced. Collapsing each relator face along a free cancellation of its word leaves a sphere in which the outer face and every child face still have embedded boundary, and the complement of the open outer face is a closed disc cellulated by the child faces.*

*Proof.* A free reduction of a closed word to the empty word pairs its sides without crossings, one adjacent inverse pair at a time; the paired sides form a plane tree whose contour is the word, each edge traversed once in each direction. Replacing the relator disc by this tree leaves the sphere a sphere: a regular neighbourhood of the tree is a disc whose boundary circle retracts onto the tree along the contour, and the complementary disc, attached along that circle, together with the tree is again a sphere. Each paired unit step is one segment with opposite traversal, so the placed boundary map factors through the tree. The closed relator discs are pairwise disjoint by Lemma B.9, so the six replacements are independent. A relator meeting the outer face in at most one vertex identifies no two outer points. In the remaining case the intersection is a subpath of two consecutive sides of the outer word, which is reduced, and a reduced walk in a tree is injective, since between two visits

to one vertex it would reverse an edge, and reversal of a tree edge reverses its unit-step label; so the subpath maps injectively into the tree. The outer boundary therefore remains embedded, and so does each child boundary, since a cancellation identifies only points with the same placed image and the child boundary is simple. An embedded circle in the sphere bounds a disc on either side; the side complementary to the outer face is the disc asserted, and its faces are the children.  $\square$

**Lemma B.11** (Planar realization). *Let  $D$  be a closed disc with a finite cellulation by closed polygonal discs, and let  $\varphi: D \rightarrow \mathbb{C}$  be continuous, an orientation-preserving homeomorphism from each face onto a polygonal disc, and linear from each edge onto a unit segment, the images of distinct faces having disjoint interiors. If  $R = \varphi(D)$  is a closed topological disc and the boundary walk  $W = \varphi|_{\partial D}$  is cyclically reduced, then  $\varphi$  is a homeomorphism onto  $R$ .*

*Proof.* Along an internal edge the two incident face images lie on opposite sides, their interiors being disjoint. At an internal vertex the sectors of the faces around it follow one another, successive sectors sharing the image of their common edge with opposite traversal, so their angles sum to a positive multiple of  $2\pi$ , and disjointness bounds the sum by  $2\pi$ : the sectors fill a neighbourhood of the image point exactly once, and no other face meets that neighbourhood. With injectivity on each closed face this shows that  $\varphi$  is injective on the interior of  $D$ , a local homeomorphism there, and that no boundary point of  $D$  has the image of an interior point.

Let  $G = \varphi(\partial D)$ , an embedded plane graph once its finitely many segments are subdivided at their intersections. Surjectivity gives  $R \setminus G = \varphi(\text{int } D)$ , a connected open set, and  $\partial R \subset G$ , an interior point of  $D$  mapping to an interior point of  $R$ . The complement of  $G$  in the plane therefore has exactly two components,  $\varphi(\text{int } D)$  and the exterior of  $R$ , so by Euler's formula the connected plane graph  $G$  has exactly one independent cycle. That cycle is the simple closed curve  $\partial R$ , and every remaining edge of  $G$  lies in a tree attached to it at one vertex. Such a tree, if nonempty, has a leaf off  $\partial R$ ; the walk  $W$  covers  $G$ , a leaf cannot lie in the interior of a unit step, and at a visit to the leaf the next unit step reverses the preceding one, contrary to reducedness. Hence  $G = \partial R$ , and  $W$  is a reduced closed walk on this cycle, so it runs in one direction and makes an integral number of circuits. The oriented boundaries of the faces of  $D$  cancel on internal edges, and at a point  $z \in \text{int } R$  off the face boundaries their winding numbers sum to one, the face interiors being disjoint; so  $\text{wind}(W, z) = 1$ , the orientation is positive and the number of circuits is one. Thus  $\varphi$  is injective on  $\partial D$ , with image disjoint from that of the interior, and a continuous bijection from the compact space  $D$  onto  $R$  is a homeomorphism.  $\square$

#### B.4. The exposed boundary at every level.

**Proposition B.12.** *For every positive level the recursively defined singleton cycle is a simple closed polygonal curve, and it is the exposed boundary of the canonical singleton supertile, traversed once as an ordered polygonal path with the placements (4.2), so its four paths are those of Table 2; the Mystic cycle is obtained by (4.5).*

*Proof.* Level one is Proposition B.3, whose single-cycle clause gives simplicity for both clusters. Assume the statement for both clusters through level  $m \geq 1$ . Let  $R$  be the support of either parent at level  $m + 1$ , the union of its children, which have disjoint interiors and whose union is a closed disc by Lemma 3.2, with placements (4.2) by Proposition 3.5; and let  $W$  be the candidate boundary word, in the frame in which all child boundaries are positively oriented.

Evaluate the spherical diagram of Lemma B.9 for that cluster on the singleton arcs of level  $m - 1$ . Its child faces carry the simple boundaries of level  $m$  of the inductive hypothesis, its outer face carries  $W$  with reversed orientation, and its six relator faces carry transported copies of  $\mathcal{N}_{m-1}$ ; the paired arcs agree as placed paths with opposite traversal, by Corollary B.6 and the row identities of Tables 5–6, and subdivision into unit steps preserves the abstract diagram. Lemma B.8 cancels every relator freely and Lemma B.5 makes  $W$  cyclically reduced, so Lemma B.10 gives a closed disc  $D$  cellulated by the children with boundary word  $W$ .

Map each child face onto its placed child by a homeomorphism of closed discs extending the boundary parametrization, which exists by identifying both with the unit disc and extending radially. The extensions agree on common edges, since every cancellation pairs one placed segment with opposite traversal, and define a continuous surjection  $\varphi: D \rightarrow R$ , orientation-preserving on each face and linear on each unit edge. By Lemma B.11,  $\varphi$  is a homeomorphism, so  $W$  is the frontier of the parent, traversed once in the candidate order. For the singleton its four paths are the words of Table 2; for the Mystic, omitting slot 2 gives (4.5), its first, second and fourth words agree with those of the singleton, and only its fourth path is used on the exterior of slot 7 at the next level. This completes the induction for both clusters.  $\square$

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