

COMPLETENESS OF TWO DYADIC SEQUENCES: ERDŐS PROBLEM #354

PINGYOU LTD

ABSTRACT. We show that the set of values of two rounded dyadic sequences $\lfloor 2^n \alpha \rfloor$, $\lfloor 2^n \beta \rfloor$, floors or ceilings, is strongly complete, that is, complete after every finite deletion, if and only if α/β is not an integral power of two and α, β are not both dyadic rationals. This settles Hegyvári's conjecture in its strongest form and answers the dyadic question for every irrational ratio; with one sequence rounded down and the other up, only the dyadic condition remains.

We prove that the second question, at a base $\gamma \in (1, 2)$, has both answers. At every base $2^{1/k}$ a single sequence $\lfloor t\gamma^n \rfloor$ is strongly complete for every $t > 0$. At the tetranacci base an explicit pair with irrational ratio has subset sums exactly the even integers, and at bases tending to two both terms of a pair with irrational ratio can be made divisible by any prescribed integer. Taken together, both questions are governed by the binary carries of the coefficients: where carries occur infinitely often, signed expansions lower the gaps of a represented set and continued fractions restore its width; where an algebraic base fixes the fractional parts, parity is trapped.

1. INTRODUCTION

For a set E of positive integers, write

$$\text{FS}(E) = \left\{ \sum_{a \in F} a : F \subseteq E \text{ finite} \right\},$$

including the empty sum. The set E is *complete* if $\text{FS}(E)$ contains every sufficiently large integer, and *strongly complete* if $E \setminus F$ is complete for every finite $F \subseteq E$. A *dyadic rational* is a number $m/2^r$ with $m \in \mathbb{Z}$ and $r \geq 0$.

Erdős and Graham [4, p. 58], following Graham [6, Question 12], ask whether the two sequences $\lfloor 2^n \alpha \rfloor$ and $\lfloor 2^n \beta \rfloor$ are complete when $\alpha, \beta > 0$ and α/β is irrational: must every sufficiently large integer have a representation

$$N = \sum_{s \in S} \lfloor 2^s \alpha \rfloor + \sum_{t \in T} \lfloor 2^t \beta \rfloor, \quad S, T \subseteq \mathbb{Z}_{\geq 0} \text{ finite?} \quad (1.1)$$

They also ask what happens when 2 is replaced by a real base $\gamma \in (1, 2)$. The two questions are recorded as Problem 354 at <https://www.erdosproblems.com/354>. In (1.1) a value attained at two indices may be used twice. The results below concern the ordinary sets of values

$$\begin{aligned} E_{\alpha, \beta}^- &= \{ \lfloor 2^n \alpha \rfloor, \lfloor 2^n \beta \rfloor : n \geq 0 \} \setminus \{0\}, \\ E_{\alpha, \beta}^+ &= \{ \lceil 2^n \alpha \rceil, \lceil 2^n \beta \rceil : n \geq 0 \}, \\ E_{\alpha, \beta}^\pm &= \{ \lfloor 2^n \alpha \rfloor, \lceil 2^n \beta \rceil : n \geq 0 \} \setminus \{0\}, \end{aligned}$$

in which a repeated value is available once, and they are correspondingly stronger.

Theorem A (Equal rounding). *For $\alpha, \beta > 0$, each of $E_{\alpha, \beta}^-$ and $E_{\alpha, \beta}^+$ is strongly complete if and only if*

$$\frac{\alpha}{\beta} \notin 2^{\mathbb{Z}} \quad \text{and} \quad \alpha, \beta \text{ are not both dyadic rationals.} \quad (1.2)$$

In particular, if α/β is irrational then, for every $B_0 \geq 0$, every sufficiently large integer has a representation (1.1) with all summands distinct and greater than B_0 ; the same holds with ceilings in place of floors.

The first question therefore has an affirmative answer. Hegyvári [8] conjectured that the hypothesis of irrational ratio can be weakened to (1.2); [Theorem A](#) proves this, and proves that (1.2) is also necessary, for ordinary value sets and after every finite deletion. The power-of-two exclusion cannot be dropped: when $\beta = 2^k\alpha$ the two equally rounded sequences have a common tail, and a single rounded dyadic sequence is never strongly complete ([Proposition 2.5](#)). Opposite rounding supplies adjacent values and removes this obstruction.

Theorem B (Opposite rounding). *For $\alpha, \beta > 0$, the set $E_{\alpha, \beta}^{\pm}$ is strongly complete if and only if α, β are not both dyadic rationals.*

Interchanging the coefficients gives the statement with the ceiling on the first sequence.

For the second question let γ be the positive root of $x^4 - x^3 - x^2 - x - 1$, and for $m \geq 4$ let $\gamma_m \in (1, 2)$ be the positive root of $x^m - x^{m-1} - \dots - x - 1$, so that $\gamma_4 = \gamma$ and $\gamma_m \rightarrow 2$; the positive constants c_m are defined in (8.5).

Theorem C (Bases between one and two). (i) *Let $k \geq 2$. For every $t > 0$ the sets $\{\lfloor t2^{n/k} \rfloor : n \geq 0\} \setminus \{0\}$ and $\{\lceil t2^{n/k} \rceil : n \geq 0\}$ are strongly complete. Consequently the union of two floor rays, or of two ceiling rays, at the base $2^{1/k}$ is strongly complete for every pair of positive coefficients.*
(ii) *Put $\alpha = 2\gamma/(5\gamma - 8)$ and $\beta = \gamma\alpha$. Then $\alpha, \beta > 0$, α/β is irrational, and the finite subset sums of the indexed multiset $(\lfloor \alpha\gamma^n \rfloor, \lfloor \beta\gamma^n \rfloor)_{n \geq 0}$, each occurrence usable once, are exactly the nonnegative even integers.*
(iii) *Let $m \geq 4$ and $2 \leq d < m - 1$. There is an integer $N \geq 0$ such that, for $\alpha = dc_m\gamma_m^N$ and $\beta = \gamma_m\alpha$, every term $\lfloor \alpha\gamma_m^n \rfloor$ and $\lfloor \beta\gamma_m^n \rfloor$ is divisible by d ; the ratio β/α is irrational, and the multiset $(\lfloor \alpha\gamma_m^n \rfloor, \lfloor \beta\gamma_m^n \rfloor)_{n \geq 0}$ is not complete.*

Thus the assertion for arbitrary bases in (1, 2) is false, with the multiplicities of the original question and already for the first sequence alone, which represents every even integer in (ii); and it is true at every base $2^{1/k}$, where one rounded ray suffices. Section 8 also gives a pair at the tetranacci base whose coefficient ratio lies outside $\mathbb{Q}\gamma^{\mathbb{Z}}$ ([Corollary 8.4](#)) and extends (iii) to polynomial multipliers ([Corollary 8.6](#)).

The proofs of [Theorems A](#) and [B](#) keep a finite set of represented integers with consecutive gaps at most g , a *represented g -band*, whose width is at least twice the largest term used. The binary recurrence lets every later term be adjoined without increasing the gap bound, so a represented 1-band of this kind proves completeness. To lower g , signed binary expansions of the two rays produce two disjoint blocks of terms whose sums differ by one; translating the band by both sums reduces the gap bound by one but not the width, and a rectangle of subset sums of later, unused terms restores the width. For irrational ratio the blocks and the rectangle are taken at two continued-fraction scales chosen together, with no bound on partial quotients and no frequency estimate for carries. For rational ratio the restoring sums lie near an exact lattice, and a vacant binary digit supplies a finer mesh where the lattice is too coarse. Part (i) of [Theorem C](#) uses two residue classes of exponents, which form two dyadic rays with irrational ratio; parts (ii) and (iii) rest on the integrality of the trace $\sum_u c_u u^m$ over the conjugates of the base.

Sections 2 and 3 set up bands and carry intervals, Sections 4 and 5 prove [Theorem A](#), Section 6 proves [Theorem B](#), Section 7 shows that the two blocks and the restoring rectangle cannot be drawn from independent supplies, Section 8 proves [Theorem C](#), and Appendix A records exact thresholds and a finite restoration.

Hegyvári [7] proved completeness in (1.1) when one coefficient is a dyadic rational and the other is not, and failure for $\beta = 2^k\alpha$ when $\alpha \geq 2$; for $1 < \alpha < 2$, Jiang and Ma [10] and Fang and He [5] proved failure for all sufficiently large k . Hegyvári also proved that, for fixed α , the set of β giving an incomplete multiset has measure zero or infinite measure [8], and constructed continuum many pairs whose subset sums contain no infinite arithmetic progression [9]; Chen and Fang [2] treat single

sequences $\lfloor \alpha s_n \rfloor$, and Bergelson and Simmons [1] strong completeness for broader classes of sets. The discussion on the problem's page records van Doorn's proofs that the floor multiset is complete for $\alpha < 2 < \beta < 3$ and that the ceiling multiset is complete whenever at least one coefficient is not a dyadic rational. Two unreferenced claims posted there bear on [Theorems A and C](#). Yu and Chen (github.com/Andrewyzzz/erdos354, 13 September 2026, AI-assisted) claim strong completeness of the two floor sequences for every irrational ratio, the irrational case of the sufficiency in [Theorem A](#), by descent of a modular gap invariant; the classification, the ceiling and mixed-rounding cases and the results at other bases are not addressed there. Geneson (arXiv:2609.25107, 20 September 2026) gives a negative answer to the second question at a base in $(6/5, 13/10)$ by combining a theorem of Dubickas [3] with a sign adjustment; [Theorem C\(ii\)](#) gives an explicit pair at the tetranacci base with its subset-sum set determined exactly. The use of binary carries continues a feature of Hegyvári's argument [7]; the additional step here is the restoration of width after each gap reduction, chosen jointly with the unit that reduces it.

2. REPRESENTED BANDS

Indices are nonnegative integers, an integer interval $[u, v]$ means $[u, v] \cap \mathbb{Z}$, and the width of a finite nonempty set is its maximum minus its minimum. We use $\{x\} = x - \lfloor x \rfloor$, $v_2(K)$ for the 2-adic valuation of K , and $\text{bit}_i(K)$ for its binary digit at 2^i . Implied constants may depend on the coefficients and on the band at the beginning of a round, but not on the scale chosen within that round.

Definition 2.1. Let $\mathcal{E}$ be a finite set of positive integers, called the *available terms*, and let $g \geq 1$ be an integer. A *represented g -band* on $\mathcal{E}$ is a nonempty finite set $X \subseteq \text{FS}(\mathcal{E})$ whose consecutive elements differ by at most g . Different elements of X may use different subsets of $\mathcal{E}$. Its *spanning interval* is $[\min X, \max X]$, and its *width* is

$$\text{width}(X) = \max X - \min X.$$

Lemma 2.2 (Union). *Let $X_1, \dots, X_r$ be represented g -bands on the same set of available terms, ordered so that $\min X_1 \leq \dots \leq \min X_r$. If*

$$\min X_{i+1} \leq \max(X_1 \cup \dots \cup X_i) + g \quad (1 \leq i < r),$$

then $X_1 \cup \dots \cup X_r$ is a represented g -band.

Proof. For a finite nonempty set $X \subset \mathbb{R}$, consecutive gaps are at most g precisely when the real intervals $[x, x + g]$, $x \in X$, have union $[\min X, \max X + g]$. The displayed inequalities ensure that the corresponding intervals for $X_1, \dots, X_r$ join successively into one interval. It follows that $X_1 \cup \dots \cup X_r$ has gaps at most g , and each of its elements retains its original representation. $\square$

2.1. Two sequences. Let $\alpha, \beta > 0$. Choose floors or ceilings for each sequence and put

$$a_j = \lfloor 2^j \alpha \rfloor \quad \text{or} \quad a_j = \lceil 2^j \alpha \rceil, \quad b_j = \lfloor 2^j \beta \rfloor \quad \text{or} \quad b_j = \lceil 2^j \beta \rceil,$$

with the chosen convention fixed along each sequence. Assume that all terms are positive and pairwise distinct as integers. Write

$$S_m = \sum_{j=0}^m (a_j + b_j), \quad H_m = \max(a_m, b_m). \quad (2.1)$$

Either rounding convention gives

$$2a_j - 1 \leq a_{j+1} \leq 2a_j + 1, \quad 2b_j - 1 \leq b_{j+1} \leq 2b_j + 1. \quad (2.2)$$

Lemma 2.3 (Continuation). *Let X be a represented g -band on the terms through level m , with width $W \geq 2H_m$. Adjoin the later terms one at a time, replacing the current set Y by $Y \cup (Y + w)$ when the term w is adjoined. After level $k \geq m$, the result is a represented g -band with minimum $\min X$ and width $S_k - \delta$, where $\delta = S_m - W \geq 0$. Thus, immediately before level $n > m$ is adjoined, its width is*

$$W_n = S_{n-1} - \delta = (\alpha + \beta)2^n + O(n). \quad (2.3)$$

An initial band satisfying $W \geq 2H_m$ exists for some m and some finite g .

Proof. Suppose Y has width V , and let w be a positive term unused in its representations. If $w \leq V + 1$, then

$$\min(Y + w) = \min Y + w \leq \max Y + 1 \leq \max Y + g.$$

The union lemma shows that $Y \cup (Y + w)$ is a represented g -band. Its minimum is unchanged and its width is $V + w$.

Now put $H = H_m$. Both terms of level $m+1$ are at most $2H+1 \leq W+1$, so they can be adjoined successively. If $l_{m+1} = \min(a_{m+1}, b_{m+1})$, then $l_{m+1} \geq 1$ and $H_{m+1} \leq 2H+1$. Consequently the new width satisfies

$$W + a_{m+1} + b_{m+1} - 2H_{m+1} = W + l_{m+1} - H_{m+1} \geq 2H + 1 - H_{m+1} \geq 0.$$

The inequality needed for continuation is therefore preserved at every level. After level k the width is $W + S_k - S_m = S_k - \delta$. Every represented sum lies in $[0, S_m]$ initially, so $\delta \geq 0$; and

$$S_{n-1} = (\alpha + \beta)(2^n - 1) + O(n)$$

gives (2.3).

For the initial band, take $X = \text{FS}(\{a_j, b_j : j \leq m\})$. It has width S_m and some finite gap bound. Since

$$\frac{S_m}{H_m} \rightarrow \frac{2(\alpha + \beta)}{\max(\alpha, \beta)} > 2,$$

it satisfies the required width inequality for all sufficiently large m . $\square$

Lemma 2.4 (Gap reduction). *Let $g \geq 2$ and let X be a represented g -band. Suppose Z_0, Z_1 are disjoint sets of available terms, disjoint also from the terms used to represent X , and*

$$\sum Z_0 = z, \quad \sum Z_1 = z + 1.$$

Then

$$Y = (X + z) \cup (X + z + 1)$$

is a represented $(g-1)$ -band of width $\text{width}(X) + 1$. Each representation consists of a subset representing an element of X , together with Z_0 or Z_1 .

Proof. Translate by $-z$. For consecutive elements $x < x'$ of X , the new set $X \cup (X + 1)$ contains $x + 1$. A gap of length $d \geq 2$ is therefore divided into gaps of lengths at most 1 and $d - 1$; gaps of length one remain harmless. The new endpoints are $\min X$ and $\max X + 1$, so all gaps are at most $g - 1$ and the width increases by one. The disjointness assumptions give the asserted representations. $\square$

2.2. Normalisation. Suppose $\alpha/\beta \notin 2^{\mathbb{Z}}$. Deleting an initial segment of a sequence multiplies its coefficient by a power of two. Such deletions, together with interchanging the sequences, arrange that

$$1 < \rho := \beta/\alpha < 2. \quad (2.4)$$

The quantities $2^j(\beta - \alpha)$ and $2^j(2\alpha - \beta)$ then tend to infinity. After a further common deletion, either choice of rounding gives

$$a_j < b_j < a_{j+1} \quad (j \geq 0). \quad (2.5)$$

Thus all terms are distinct as integers. The deletion can also place every remaining term above any prescribed finite set. To prove strong completeness of $E_{\alpha,\beta}^-$ or $E_{\alpha,\beta}^+$, it consequently suffices to prove completeness after these deletions. For rational ratios we shall instead use coefficients given by coprime odd integers times a common scale; the distinctness argument for that normalisation is given in [Section 5](#).

2.3. The necessary conditions.

Proposition 2.5. *Let $\alpha, \beta > 0$.*

- (a) *If both coefficients are dyadic rationals, none of $E_{\alpha,\beta}^-$, $E_{\alpha,\beta}^+$ and $E_{\alpha,\beta}^\pm$ is strongly complete.*
- (b) *If $\alpha/\beta \in 2^\mathbb{Z}$, neither $E_{\alpha,\beta}^-$ nor $E_{\alpha,\beta}^+$ is strongly complete.*

Proof. (a) If $\alpha = m/2^r$, then $2^n\alpha$ is an even integer for $n > r$; the same holds for β . Every set in (a) therefore has only finitely many odd elements. Deleting them leaves only even subset sums.

(b) Interchanging the coefficients if necessary, write $\beta = 2^k\alpha$ with $k \geq 0$. The second floor sequence is a tail of the first, and the same is true for ceilings. After a finite deletion the ordinary value set is therefore $\{u_j : j \geq 0\}$, where

$$u_j = \lfloor \xi 2^j \rfloor \quad \text{or} \quad u_j = \lceil \xi 2^j \rceil \quad \text{for some } \xi \geq 2.$$

We show that neither tail is complete.

For floors, write $u_{j+1} = 2u_j + \varepsilon_j$ with $\varepsilon_j \in \{0, 1\}$. Induction gives

$$u_n - \sum_{j < n} u_j = u_0 + \sum_{j < n} \varepsilon_j \geq 2.$$

Hence $1 + \sum_{j < n} u_j$ is larger than every sum using only earlier terms and smaller than every term of index at least n . It is not represented, and these omitted integers tend to infinity.

For ceilings, $u_n \geq \xi 2^n \geq 2^{n+1}$. A subset sum below u_n uses only the n preceding terms, so at most 2^n nonnegative integers below u_n are represented. At least $u_n - 2^n \geq 2^n$ are omitted. Their number is unbounded, which excludes completeness. $\square$

3. SIGNED BINARY EXPANSIONS

For $y > 0$ and $i \geq 0$, put

$$u_i^-(y) = \lfloor 2^i y \rfloor, \quad u_i^+(y) = \lceil 2^i y \rceil, \quad F_y^\pm(K) = \sum_{i \geq 0} \text{bit}_i(K) u_i^\pm(y) \quad (K \geq 1).$$

Thus $F_y^\pm(K)$ is the rounded sum selected by the ordinary binary digits of K . Define the floor and ceiling carries by

$$\varepsilon_i^-(y) = u_{i+1}^-(y) - 2u_i^-(y), \quad \varepsilon_i^+(y) = 2u_i^+(y) - u_{i+1}^+(y). \quad (3.1)$$

Both belong to $\{0, 1\}$. For either rounding convention, eventual vanishing of the carries is equivalent to y being dyadic rational. Indeed, vanishing from index i_0 gives $u_i^\pm(y) = 2^{i-i_0} u_{i_0}^\pm(y)$ for every $i \geq i_0$; division by 2^i and passage to the limit yield $y = 2^{-i_0} u_{i_0}^\pm(y)$. The converse follows directly when $y = m/2^r$. In particular, a nondyadic coefficient has infinitely many nonzero carries for each convention.

Lemma 3.1 (Signed carry intervals). *Let $K, H \geq 1$ be integers with $2^H > K$, and put $v = v_2(K)$. Among the signed expansions*

$$K = \sum_{i=v}^H \sigma_i 2^i, \quad \sigma_i \in \{-1, 0, 1\},$$

the rounded sums $\sum_{i=v}^H \sigma_i u_i^\pm(y)$ realize every integer of the respective intervals

$$I_{K,H}^-(y) = [F_y^-(K), u_H^-(y) - F_y^-(2^H - K)], \quad (3.2)$$

$$I_{K,H}^+(y) = [u_H^+(y) - F_y^+(2^H - K), F_y^+(K)]. \quad (3.3)$$

They contain $\lfloor Ky \rfloor$ and $\lceil Ky \rceil$, respectively, and have widths

$$\text{width } I_{K,H}^\pm(y) = \sum_{i=v}^{H-1} \varepsilon_i^\pm(y). \quad (3.4)$$

The ordinary binary expansion realizes the lower floor endpoint and the upper ceiling endpoint, using only the nonzero digit positions of K .

Proof. First suppose K is odd. Write $K = \sum_{i=0}^H k_i 2^i$, with $k_0 = 1$ and $k_H = 0$. For bits $c_1, \dots, c_H$, set $c_0 = c_{H+1} = 0$ and

$$\sigma_i = k_i + c_i - 2c_{i+1} \quad (0 \leq i \leq H).$$

The weighted sum telescopes to K . For $1 \leq i < H$, the digit σ_i belongs to $\{-1, 0, 1\}$ exactly when

$$c_{i+1} \leq c_i \quad \text{if } k_i = 0, \quad c_i \leq c_{i+1} \quad \text{if } k_i = 1.$$

There is no additional restriction at $i = 0$ or $i = H$. These relations orient the path on $c_1, \dots, c_H$. Starting with all coordinates zero, set them to one in an order that respects the indicated predecessors. Such an order exists because an oriented path has no directed cycle. Every intermediate vector is admissible.

The rounded sums along this path are

$$\begin{aligned} \sum_i \sigma_i u_i^-(y) &= F_y^-(K) + \sum_{j=1}^H c_j \varepsilon_{j-1}^-(y), \\ \sum_i \sigma_i u_i^+(y) &= F_y^+(K) - \sum_{j=1}^H c_j \varepsilon_{j-1}^+(y). \end{aligned}$$

Each step therefore changes the value by zero or one, monotonically in the appropriate direction. Every integer between the endpoints occurs.

At the all-ones endpoint the signed digits are $\sigma_0 = -1$, $\sigma_i = k_i - 1$ for $1 \leq i < H$, and $\sigma_H = 1$. The resulting exact identity is

$$K = 2^H - 1 - \sum_{i=1}^{H-1} (1 - k_i) 2^i.$$

The expression $1 + \sum_{i=1}^{H-1} (1 - k_i) 2^i$ is the ordinary binary expansion of $2^H - K$. Hence the rounded endpoint is $u_H^\pm(y) - F_y^\pm(2^H - K)$, and its distance from the ordinary endpoint is the sum of the carries. This proves the interval and width formulas for odd K . For $K = 2^v k$ with k odd, apply the same argument to k , $2^v y$, and $H - v$; insert zero digits below v .

It remains to locate the rounded multiple. A sum of floors is at most the floor of the sum, so $F_y^-(K) \leq \lfloor Ky \rfloor$, while

$$u_H^-(y) - F_y^-(2^H - K) \geq \lfloor 2^H y \rfloor - \lfloor (2^H - K)y \rfloor \geq \lfloor Ky \rfloor.$$

For ceilings, $F_y^+(K) \geq \lceil Ky \rceil$ and

$$u_H^+(y) - F_y^+(2^H - K) \leq \lceil 2^H y \rceil - \lceil (2^H - K)y \rceil \leq \lceil Ky \rceil.$$

These establish the asserted containment. $\square$

For example, take $K = 3$, $H = 3$, and $y = \sqrt{2}$. The admissible carry vectors

$$(0, 0, 0), \quad (0, 1, 0), \quad (1, 1, 0), \quad (1, 1, 1)$$

give the signed expansions $3 = 1 + 2 = 1 - 2 + 4 = -1 + 4 = -1 - 4 + 8$. Since $(u_0^-, u_1^-, u_2^-, u_3^-) = (1, 2, 5, 11)$, their rounded values are 3, 4, 4, 5. Thus every integer of $I_{3,3}^-(\sqrt{2}) = [3, 5]$ occurs; the zero increment reflects a vanishing carry.

Write

$$h(K) = \lfloor \log_2 K \rfloor + 1, \quad I_K^\pm(y) = I_{K,h(K)}^\pm(y).$$

The top index $h(K)$ is the least allowed in the lemma. An integer interval is *nondegenerate* if it contains at least two integers.

Lemma 3.2 (A unit). *Let P, Q be positive integers and let $e = |Q\alpha - P\beta| > 0$. Suppose $n_0 \geq 0$ satisfies $1/2 \leq 2^{n_0}e < 1$, and suppose the two rays have distinct integer values and use the same rounding convention. For some $n \in \{n_0, n_0 + 1\}$ there are disjoint blocks Z_0, Z_1 of terms, supported in $[n, n + h]$ with $h = h(\max(P, Q))$, such that $\sum Z_1 = \sum Z_0 + 1$. Moreover*

$$\frac{1}{2e} \leq 2^n < \frac{2}{e}. \quad (3.5)$$

Proof. Put $y = 2^{n_0}\alpha$ and $z = 2^{n_0}\beta$. For floors, the intervals $I_Q^-(y)$ and $I_P^-(z)$ contain $\lfloor Qy \rfloor$ and $\lfloor Pz \rfloor$. These integers are equal or adjacent because $|Qy - Pz| < 1$. If adjacent, use their signed realizations. If equal and either interval is nondegenerate, the intervals again contain adjacent integers, one in each. The only exception is a common singleton $\{\kappa\}$. Then write $Qy = \kappa + u$ and $Pz = \kappa + v$, with $u, v \in [0, 1)$. Since $|u - v| \geq 1/2$, $\lfloor 2u \rfloor$ and $\lfloor 2v \rfloor$ differ by one. Therefore the anchors of $I_Q^-(2y)$ and $I_P^-(2z)$ at the next level are adjacent.

For ceilings the same argument starts from $\lceil Qy \rceil$ and $\lceil Pz \rceil$. In the common-singleton case, write $Qy = \kappa - u$ and $Pz = \kappa - v$, with $u, v \in [0, 1)$. The next anchors are $2\kappa - \lfloor 2u \rfloor$ and $2\kappa - \lfloor 2v \rfloor$, which again differ by one.

In either convention, select the resulting adjacent signed sums and subtract the smaller from the larger. Move every negative term to the other side. Each term has coefficient in $\{-1, 0, 1\}$ and occurs in at most one ray, so the positive and negative supports are disjoint subsets of the available terms. They are the required blocks Z_1, Z_0 . Their indices lie in $[n, n + h]$, and the choice $n \in \{n_0, n_0 + 1\}$ gives (3.5). $\square$

4. IRRATIONAL RATIOS

Assume throughout that the rays have the same rounding convention and have been normalised so that $1 < \rho = \beta/\alpha < 2$ and $a_j < b_j < a_{j+1}$, with ρ irrational. The unit lemma reduces the gap bound; later terms restore the continuation inequality. The unit and the restoring translates must be chosen at compatible scales.

4.1. Two scales. After omitting any initial repeated denominator, let Q_k/P_k be the continued-fraction convergents of ρ , with strictly increasing positive denominators. They satisfy

$$\frac{1}{P_{k+1} + P_k} < |P_k\rho - Q_k| < \frac{1}{P_{k+1}}, \quad (4.1)$$

and $\gcd(P_k, Q_k) = 1$: if $\tau = a_{k+1} + 1/\tau'$ with $\tau' > 1$ is the complete quotient following a_{k+1} , then $\rho = (Q_k\tau + Q_{k-1})/(P_k\tau + P_{k-1})$ and $|P_k\rho - Q_k| = 1/(P_k\tau + P_{k-1}) = 1/(P_{k+1} + P_k/\tau')$. In particular $|\rho - Q_k/P_k| < 1/P_k^2$.

Let Q/P be a sufficiently large convergent with successor denominator P^+ , and apply Lemma 3.2 with $e = |Q\alpha - P\beta| = \alpha|P\rho - Q|$. By (4.1) and (3.5), the level n of the unit satisfies

$$\frac{P^+}{2\alpha} < 2^n < \frac{4P^+}{\alpha}. \quad (4.2)$$

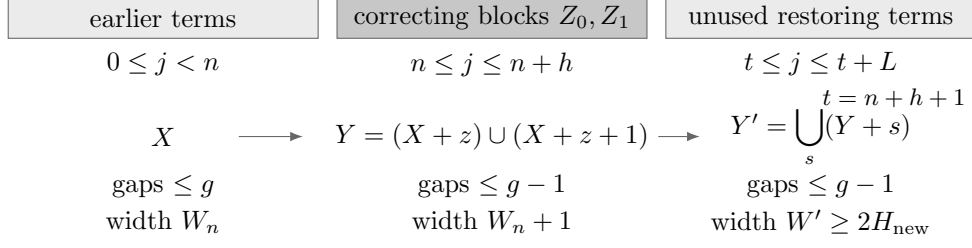


FIGURE 1. One round of gap reduction and restoration for an irrational ratio. The sums of Z_0 and Z_1 are z and $z + 1$, each s is a subset sum of the restoring terms, and H_{new} is the largest term at the final level. In every representation the three support blocks are disjoint; overlapping translates preserve the reduced gap bound, and the restored width permits continuation.

Lemma 4.1 (Joint choice of scales). *There are convergents Q/P of ρ with P arbitrarily large, later convergent denominators q , and levels n of the unit of Lemma 3.2 for Q/P , such that*

$$q \geq 128P, \quad \frac{\alpha}{4} 2^n < q < 256\alpha 2^n. \quad (4.3)$$

In particular $\log q = O(n)$ along these choices.

Proof. Let q be a large convergent denominator and let P be the last convergent denominator with $P \leq q/128$; its successor P^+ satisfies $q/128 < P^+ \leq q$. By (4.2),

$$\frac{q}{256\alpha} < \frac{P^+}{2\alpha} < 2^n < \frac{4P^+}{\alpha} \leq \frac{4q}{\alpha}.$$

Both P and n tend to infinity with q ; no bound on P^+/P is needed. $\square$

Lemma 4.2 (A rectangle of nominal sums). *Let $A > 0$, $B = \rho A$, and let q be a convergent denominator of ρ . For an integer $K \geq 16q$ put $m_0 = \lceil \rho(q - 1) \rceil$. The set $\{Au + Bv : u, v \in \mathbb{Z}, 0 \leq u, v \leq K\}$ contains a subset with least element Am_0 , greatest element $AK + B(K - q + 1)$, consecutive gaps at most $3A/q$, and hence span at least*

$$A((1 + \rho)K - 4q - 1). \quad (4.4)$$

Proof. Let p/q be the convergent. The points vp/q , $0 \leq v < q$, are the full grid of mesh $1/q$ on $\mathbb{R}/\mathbb{Z}$, and $|vp - vp/q| < v/q^2 < 1/q$, so the points $\{vp\}$, $0 \leq v < q$, leave no arc of $\mathbb{R}/\mathbb{Z}$ of length greater than $3/q$ empty.

Work in units of A . For $m_0 \leq m \leq K - 1$ and $0 \leq v < q$ put $u = m - \lfloor \rho v \rfloor$; then $0 \leq u \leq K$ and $u + \rho v = m + \{\rho v\}$. Together with the point K (from $u = K, v = 0$) these points cover $[m_0, K]$ with consecutive gaps at most $3/q$. Now translate this set by ρv_0 for $0 \leq v_0 \leq K - q + 1$: the second coordinate becomes $v + v_0 \leq K$, so the translates stay in the rectangle, and the spanning intervals of successive translates overlap because $K - m_0 > \rho$. The union has least element m_0 , greatest element $K + \rho(K - q + 1)$, gaps at most $3/q$, and span at least $(1 + \rho)K - 2\rho(q - 1) - 1 \geq (1 + \rho)K - 4q - 1$. $\square$

Lemma 4.3 (Stability under rounding). *Let $x_1 < \dots < x_r$ be real numbers with consecutive differences at most d , and let $y_1, \dots, y_r$ satisfy $|y_i - x_i| \leq D$. Then the distinct values among the y_i , in increasing order, have consecutive differences at most $d + 2D$ and span at least $x_r - x_1 - 2D$.*

Proof. Suppose $a < b$ are consecutive distinct values among the y_i with $b - a > d + 2D$. Some x_j is at most $a + D$ (the one corresponding to a) and some x_k is at least $b - D$ (the one corresponding to b); let x_l be the first x exceeding $a + D$. Then $x_l \leq a + D + d < b - D$, so $y_l \in (a, b)$, a contradiction. The endpoints give the span. $\square$

4.2. One round.

Theorem 4.4 (Irrational ratios). *If $\alpha, \beta > 0$ and α/β is irrational, then $E_{\alpha, \beta}^-$ and $E_{\alpha, \beta}^+$ are strongly complete.*

Proof. Fix an arbitrary finite set of values to be deleted. The normalisation in Section 2.2 gives positive, pairwise distinct remaining terms, all larger than the deleted values. By Lemma 2.3, an initial prefix supports a represented g -band X of width at least $2H_m$. If $g = 1$, continuation represents every integer from $\min X$ onward. It therefore suffices to replace $g \geq 2$ by $g - 1$ while restoring the same width condition.

Gap reduction. Choose q, P, Q, n by Lemma 4.1, with q large enough that $n > m$ and all the eventual inequalities below hold. Continue X through level $n - 1$, retaining the name X . Its width is $W_n = (\alpha + \beta)2^n + O(n)$. By Lemma 3.2, disjoint blocks Z_0, Z_1 supported in $[n, n + h]$, where $h = h(\max(P, Q))$, have sums $z, z + 1$. Thus

$$Y = (X + z) \cup (X + z + 1)$$

is a represented $(g - 1)$ -band of width $W_n + 1$ by Lemma 2.4. For the restoring terms put

$$t = n + h + 1, \quad A = \alpha 2^t, \quad B = \beta 2^t = \rho A.$$

All terms of index at least t are unused. Let L be least with $K := 2^{L+1} - 1 \geq 16q$; then $K < 32q + 1$ and $L = O(\log q) = O(n)$.

Overlapping translates. For $0 \leq u, v \leq K$, their ordinary binary expansions define the actual subset sum

$$s(u, v) = \sum_{i=0}^L \text{bit}_i(u) a_{t+i} + \sum_{i=0}^L \text{bit}_i(v) b_{t+i}.$$

Each of its at most $2(L + 1)$ terms differs by less than one from the corresponding real term. With $D = 2(L + 1)$,

$$|s(u, v) - (Au + Bv)| \leq D. \quad (4.5)$$

Apply Lemma 4.2 to A, B, q and K , and realise each point $Au + Bv$ of the subset it provides by $s(u, v)$. For large P the convergent satisfies $1 < Q/P < 2$, so $2^{h+1} \leq 4 \max(P, Q) < 8P$, and with $q \geq 128P$,

$$\frac{3A}{q} = \frac{3\alpha 2^{h+1}}{q} 2^n < \frac{3\alpha}{16} 2^n.$$

By Lemma 4.3 the distinct values of $s(u, v)$ so obtained have consecutive gaps at most

$$\frac{3A}{q} + 2D < W_n + 1 \quad (4.6)$$

for q large, since $W_n = (\alpha + \beta)2^n + O(n)$ and $D = O(n)$. The translates $Y + s(u, v)$ therefore satisfy the hypothesis of Lemma 2.2, and their union Y' is a represented $(g - 1)$ -band.

Restoring the width. The largest available term through the new level is b_{t+L} , and $2b_{t+L} \leq B(K + 1) + 2$, where the constant 2 is unnecessary for floors. By (4.4), the nominal span exceeds $B(K + 1)$ by at least

$$A((1 + \rho)K - 4q - 1) - \rho A(K + 1) = A(K - 4q - 1 - \rho) \geq A(12q - 3),$$

which exceeds $2D + 2$ for q large. By Lemma 4.3 the span of the values $s(u, v)$, and hence the width of Y' , exceeds $2b_{t+L}$. So Y' is a represented $(g - 1)$ -band on the terms with index at most $t + L$ satisfying the hypothesis of Lemma 2.3.

Disjointness and termination. Each representation in Y' uses three disjoint index ranges:

$$\underbrace{[0, n - 1]}_{\text{continued band}}, \quad \underbrace{[n, n + h]}_{\text{one unit block}}, \quad \underbrace{[n + h + 1, n + h + 1 + L]}_{\text{one restoring subset}}.$$

Each term occurs at most once, and numerical interleaving makes the summands distinct integers. Alternative representations may share terms. We have therefore constructed a represented $(g-1)$ -band satisfying continuation after the prescribed deletion. Repeat until $g = 1$, then continue indefinitely. This proves completeness after the prescribed deletion, for both rounding conventions. $\square$

5. RATIONAL RATIOS

Assume that α/β is rational but not a power of 2, and that α, β are not both dyadic rationals. The two nominal rays then lie on an exact lattice. The task is to leave enough of that lattice unused by the unit that lowers the gap bound.

5.1. Normalisation and a canonical unit. Write $\alpha = A_0\tau_0$ and $\beta = B_0\tau_0$ with A_0, B_0 coprime positive integers. By deleting initial segments, the powers of 2 dividing A_0 and B_0 can be absorbed into a common coefficient. Interchanging the rays if necessary gives

$$\alpha = P\tau, \quad \beta = Q\tau, \quad 1 \leq P < Q, \quad P, Q \text{ odd}, \quad \gcd(P, Q) = 1. \quad (5.1)$$

Here $P \neq Q$ because the original ratio is not a power of 2, and τ is not dyadic because the original coefficients are not both dyadic. A further common deletion makes $\tau \geq 1$ and puts every remaining term above any prescribed finite set.

For the floor proof put

$$a_n = \lfloor P2^n\tau \rfloor, \quad b_n = \lfloor Q2^n\tau \rfloor, \quad x = 2^n\tau.$$

The integers $P2^i, Q2^j$ are all distinct: equality across the two rays would force $i = j$ and $P = Q$, by oddness. Their nominal values therefore differ by at least $\tau \geq 1$, so all rounded terms are distinct as integers. The continuation lemma applies with $H_m = b_m$. An initial band exists because $S_m/b_m \rightarrow 2(P+Q)/Q > 2$. If a band with fixed deficit δ is continued through level $n-1$, its width is

$$W_n = S_{n-1} - \delta = (P+Q)x + O(n). \quad (5.2)$$

Lemma 5.1. *Let I, J be integer intervals with a common element, at least one of which is nondegenerate. They contain $u \in I, v \in J$ with $|u - v| = 1$ such that u is the lower endpoint of I or v is the lower endpoint of J . The same assertion holds with upper endpoints.*

Proof. Interchange the intervals so that $\min I \geq \min J$. Their intersection then contains $\min I$. If J is nondegenerate, it contains a neighbour of $\min I$, so take $u = \min I$. If J is a singleton, then $\min I = \min J$ and I is nondegenerate; take $v = \min J$ and $u = v + 1$. Reflection proves the upper-endpoint assertion. $\square$

At level n consider

$$I_Q^-(Px), \quad I_P^-(Qx). \quad (5.3)$$

Both intervals contain $\lfloor PQx \rfloor$. At least one of $P\tau, Q\tau$ is nondyadic: otherwise a Bézout relation for P, Q would make τ dyadic. That ray has nonzero carries at arbitrarily large indices. Choosing n at such a carry makes its interval in (5.3) nondegenerate, since both multipliers are odd. By Lemma 5.1, adjacent integers can be chosen from the two intervals with at least one of them a lower endpoint, hence realised by the ordinary binary expansion of its multiplier.

When $P = 1$, we require the expansion on the second ray to be ordinary. This is always possible: choose n at a nonzero carry of the first ray, whose coefficient τ is nondyadic. Then $I_Q^-(x)$ is nondegenerate and contains a neighbour of $\lfloor Qx \rfloor = F_{Qx}^-(1)$.

Subtracting the selected signed expansions and moving negative terms across produces disjoint blocks Z_0, Z_1 with sums $z, z+1$, exactly as in Lemma 3.2. Thus, for a represented g -band X continued through level $n-1$, with $g \geq 2$,

$$Y = (X + z) \cup (X + z + 1) \quad (5.4)$$

is a represented $(g - 1)$ -band of width $W_n + 1$. The ordinary expansion is retained as part of the construction: its unused digits will supply the restoration.

5.2. Unused tails. Put $N = 2^{h(P)}$ and $M = 2^{h(Q)}$, the least powers of 2 strictly exceeding P, Q . A signed expansion with multiplier K uses indices at most $h(K)$ above n , whereas its ordinary expansion uses only indices below $h(K)$. Consequently the following two rows describe tails unused by both blocks:

ordinary expansion on	r_a	r_b	G	(5.5)
second ray, multiplier P	$h(Q) + 1$	$h(P)$	N	
first ray, multiplier Q	$h(Q)$	$h(P) + 1$	$\min(M, 2N)$.	

The available tails begin at $n + r_a, n + r_b$. In either row set

$$G = \gcd(P2^{r_a}, Q2^{r_b}) = 2^{\min(r_a, r_b)}, \quad c = P2^{r_a}/G, \quad d = Q2^{r_b}/G.$$

Then $\gcd(c, d) = 1$.

Lemma 5.2. *Let c, d be coprime positive integers, let $A' \geq d - 1$, $B' \geq c - 1$ be integers, and put $R = \max\{c(d - 1), d(c - 1)\}$. Every integer in*

$$[R, cA' + dB' - R]$$

has the form $cu + dv$ with $0 \leq u \leq A'$ and $0 \leq v \leq B'$ integers.

Proof. Fix a residue class modulo d , and choose $u_0 \in \{0, \dots, d - 1\}$ with cu_0 in that class. The admissible values $u = u_0 + kd$ give

$$cu + dv = cu_0 + d(ck + v).$$

The integer intervals $[ck, ck + B']$ for consecutive k overlap or abut, since $B' \geq c - 1$. The first attained integer is at most $c(d - 1)$, and the last is at least $cA' + dB' - c(d - 1)$, because the largest admissible u is at least $A' - d + 1$. Every integer of the fixed class between these bounds is therefore attained. The stated interval lies within those bounds for every class. $\square$

Fix a row of (5.5). For a common last offset $L \geq \max(r_a, r_b)$, the unused tails have nominal subset sums

$$Gx(cu + dv), \quad 0 \leq u \leq A_L = 2^{L-r_a+1} - 1, \quad 0 \leq v \leq B_L = 2^{L-r_b+1} - 1.$$

Choose L sufficiently large that $A_L \geq d - 1$, $B_L \geq c - 1$, and put $Z_L = cA_L + dB_L$. For every integer $j \in [R, Z_L - R]$, fix one mask supplied by Lemma 5.2, and denote its actual sum by t_j . Each mask contains at most $2(L + 1)$ terms, so

$$t_j = Gjx + O_{P,Q,L}(1). \quad (5.6)$$

In particular, for fixed L and sufficiently large n , the t_j increase with j and have successive differences $Gx + O_{P,Q,L}(1)$. Their total nominal range is governed by

$$GZ_L = (P + Q)2^{L+1} - P2^{r_a} - Q2^{r_b}. \quad (5.7)$$

5.3. Restoring the width.

Proposition 5.3. *If $\alpha, \beta > 0$, $\alpha/\beta \in \mathbb{Q} \setminus 2^{\mathbb{Z}}$, and α, β are not both dyadic rationals, then $E_{\alpha, \beta}^-$ is strongly complete.*

Proof. Fix a finite deletion, normalize as above, and start with a represented g -band of width at least $2H_m$. If $g = 1$, continuation completes the proof. Suppose $g \geq 2$. We show that the unit construction can be completed to a represented $(g - 1)$ -band of width at least twice the largest available term. All choices of L below depend only on P, Q and the possible row of (5.5); they will be made before the late carry index n is selected.

The case $G < P + Q$. By (5.2) and (5.6), the width $W_n + 1$ of Y exceeds the consecutive differences of the t_j for sufficiently large n . Hence

$$Y' = \bigcup_{j=R}^{Z_L-R} (Y + t_j)$$

is represented $(g-1)$ -band by Lemma 2.2. Choose L so large that

$$G(Z_L - 2R) > 2Q2^L. \quad (5.8)$$

This is possible because (5.7) makes the difference equal to $2P2^L$ minus a constant independent of L . The span of the shifts is $G(Z_L - 2R)x + O_{P,Q,L}(1)$, whereas $2b_{n+L} \leq 2Q2^Lx$. Thus Y' has width greater than $2b_{n+L}$ for sufficiently large n .

This case always applies when the second ray carries the ordinary expansion. If $P \geq 3$, then $G = N < 2P < P + Q$; if $P = 1$, the prescribed ordinary expansion gives $G = N = 2 < P + Q$.

The case $G \geq P + Q$. The ordinary expansion must therefore be on the first ray and $P \geq 3$. Here $G = \min(M, 2N)$ and $M \geq N$. If $M = N$, then $G = N < 2P < P + Q$. If $M \geq 4N$, oddness gives $Q > M/2 \geq 2N \geq G$, again a contradiction. Consequently

$$M = 2N = G, \quad P > G/4, \quad G/2 < Q \leq G - P < 3G/4. \quad (5.9)$$

The binary digit of Q at $G/4$ is zero. Since the first ray carries the ordinary expansion of Q , the term

$$f = a_{n+\log_2 G-2} = \lfloor PGx/4 \rfloor \quad (5.10)$$

is unused by either unit block. Its index is at least n and strictly below $n + r_a$, so it is also unused by X and by every tail mask defining t_j .

Write $P = 4d_0 + e_0$, where $e_0 \in \{1, 3\}$. For

$$R + d_0 \leq j \leq Z_L - R - 1$$

use the shifts

$$t_j, \quad t_{j-d_0} + f, \quad t_{j+1}. \quad (5.11)$$

Their nominal values are Gjx , $G(j+e_0/4)x$, and $G(j+1)x$. For fixed L and sufficiently large n , they occur in that order, and the whole collection has consecutive differences at most $3Gx/4 + O_{P,Q,L}(1)$. By (5.9), $P + Q > 3G/4$, so these differences are smaller than $W_n + 1$. The translates of Y by (5.11) consequently form a represented $(g-1)$ -band Y' .

The first and last unmodified shifts in this collection are t_{R+d_0} and t_{Z_L-R} . Their difference is

$$G(Z_L - 2R - d_0)x + O_{P,Q,L}(1).$$

Choose L so large that the index range is nonempty and

$$G(Z_L - 2R - d_0) > 2Q2^L. \quad (5.12)$$

Again (5.7) shows that the difference is $2P2^L$ minus a constant. For sufficiently large n , the width of Y' is therefore greater than $2b_{n+L}$.

There are only two possible rows in (5.5). Choose one L large enough for all requirements of both rows, then choose a carry index $n > m$ so large that all rounding and width estimates hold. Such indices are unbounded by the unit construction. Every representation in Y' consists of a representation in X , one unit block, one unused tail mask, and possibly the unused term f . These supports are disjoint. Thus the construction restores the hypothesis of Lemma 2.3 while reducing g by one. Finitely many repetitions give $g = 1$, and continuation proves completeness after the prescribed deletion. $\square$

Proposition 5.4. *Under the hypotheses of Proposition 5.3, $E_{\alpha,\beta}^+$ is strongly complete.*

Proof. Use the same normalization and put $a_n = \lceil P2^n \tau \rceil$, $b_n = \lceil Q2^n \tau \rceil$. Numerical distinctness, continuation, and (5.2) are unchanged. The intervals $I_Q^+(Px)$ and $I_P^+(Qx)$ both contain $\lceil PQx \rceil$. Their upper endpoints are the ordinary expansions, and their widths are sums of ceiling carries. At least one ray has infinitely many nonzero ceiling carries, so the upper-endpoint version of Lemma 5.1 gives adjacent integers with one ordinary expansion. If $P = 1$, choose a nonzero carry of the first ray: $I_Q^+(x)$ then contains a neighbour of $\lceil Qx \rceil = F_{Qx}^+(1)$, keeping the second ray ordinary as required.

The unused supports in (5.5) depend only on the masks, not on the rounding convention. The tail shifts therefore satisfy (5.6), with the same uniform error bound. In the second restoration case the same zero digit leaves $f = \lceil PGx/4 \rceil$ unused. Thus the mesh estimates, disjointness, and span estimates in the floor proof all remain valid. The final comparison is now $2b_{n+L} \leq 2Q2^Lx + 2$; the strict margins in (5.8) and (5.12) absorb the additional constant for sufficiently large n . Choose L for both rows first and then the late carry index n , as before. Each round reduces g and restores continuation, proving the assertion. $\square$

Proof of Theorem A. The two conditions in (1.2) are necessary by Proposition 2.5. Conversely, suppose that the ratio is not a power of 2 and that the coefficients are not both dyadic rationals. If the ratio is irrational, Theorem 4.4 applies; if it is rational, Propositions 5.3 and 5.4 apply. For the final assertion let α/β be irrational and $B_0 \geq 0$. Deleting from $E_{\alpha,\beta}^-$ all values at most B_0 leaves a complete set, and a representation of N by distinct elements of the remaining set is a representation (1.1) with distinct summands greater than B_0 ; the ceiling case is the same. $\square$

6. ONE FLOOR AND ONE CEILING

Throughout this section the first ray is rounded down and the second up. If both coefficients are dyadic rationals, all sufficiently late terms are even, so strong completeness fails. Sufficiency in Theorem B is proved by treating irrational ratios, equal dyadic orbits, and the remaining rational ratios.

The signed intervals supplied by Lemma 3.1 are

$$I_{K,H}^-(y) = [F_y^-(K), \lfloor 2^H y \rfloor - F_y^-(2^H - K)], \quad I_{K,H}^+(z) = [\lceil 2^H z \rceil - F_z^+(2^H - K), F_z^+(K)].$$

Here $2^H > K$, and every integer in an interval has a signed realization using indices from $v_2(K)$ through H above the base level.

6.1. A unit for opposite rounding.

Lemma 6.1. *Let P, Q be positive integers and suppose $0 < e = |Q\alpha - P\beta| < 1$. Choose $n_0 \geq 0$ with $1/2 \leq 2^{n_0}e < 1$, and set $H = \lceil \log_2(P + Q) \rceil$. For some $n \in \{n_0, n_0 + 1\}$, the intervals $I_{Q,H}^-(2^n\alpha)$ and $I_{P,H}^+(2^n\beta)$ contain adjacent integers, one in each interval. Moreover*

$$\frac{1}{2e} \leq 2^n < \frac{2}{e}.$$

If the rays have distinct integer values, these adjacent realizations give disjoint blocks of terms in levels $[n, n + H]$ whose sums differ by one.

Proof. Put $m = 2^H \geq P + Q$, $y = 2^{n_0}\alpha$, $z = 2^{n_0}\beta$, $\mu = \lfloor my \rfloor$, and $\nu = \lceil mz \rceil$. For $0 \leq i \leq H$,

$$\lfloor 2^i y \rfloor = \left\lfloor \frac{\mu}{2^{H-i}} \right\rfloor \leq \frac{2^i \mu}{m}, \quad \lceil 2^i z \rceil = \left\lceil \frac{\nu}{2^{H-i}} \right\rceil \geq \frac{2^i \nu}{m}.$$

Consequently the upper floor endpoint U and lower ceiling endpoint Λ obey

$$U = \mu - F_y^-(m - Q) \geq \frac{Q\mu}{m}, \quad \Lambda = \nu - F_z^+(m - P) \leq \frac{P\nu}{m}. \quad (6.1)$$

Write $A = Qy$ and $B = Pz$. The intervals contain $\lfloor A \rfloor$ and $\lceil B \rceil$, respectively. Since $|A - B| < 1$, we have $\lceil B \rceil \geq \lfloor A \rfloor$, which excludes a separated ceiling interval to the left. A separation $\Lambda - U \geq 2$ to the right would imply $P\nu - Q\mu \geq 2m$. But $z > (\nu - 1)/m$ and $y < (\mu + 1)/m$ would then give

$$B - A > \frac{P\nu - Q\mu - P - Q}{m} \geq 2 - \frac{P + Q}{m} \geq 1,$$

a contradiction. The two integer intervals therefore supply adjacent integers unless both are the same singleton $\{\kappa\}$.

In that exceptional case their widths are zero, so every carry on their respective supports vanishes. Each ordinary expansion consequently has value 2κ at the next level. The new intervals both contain 2κ and supply adjacent integers unless both are again singletons. We show that this second exception is impossible.

At any common singleton level, one has

$$0 \leq Qy - \kappa < \frac{Q}{m}, \quad 0 \leq \kappa - Pz < \frac{P}{m}. \quad (6.2)$$

Indeed write $Q = 2^w Q'$ with Q' odd. Vanishing floor carries from w through $H - 1$ give $\{2^w y\} < 2^{w-H}$ and $\kappa = Q' \lfloor 2^w y \rfloor$, proving the first bound. Writing $P = 2^v P'$ similarly gives $\lceil 2^v z \rceil - 2^v z < 2^{v-H}$ and $\kappa = P' \lceil 2^v z \rceil$, proving the second. A common singleton at the next level would therefore force

$$0 \leq 2Qy - 2Pz < \frac{P + Q}{m} \leq 1,$$

whereas $|2Qy - 2Pz| = 2^{n_0+1}e \geq 1$.

Subtract the signed realizations of the adjacent integers and move all negative terms to the other side. Distinct values in the two rays ensure disjoint positive and negative supports. The index and scale bounds follow from the choice of n . $\square$

6.2. Irrational ratios. Suppose α/β is irrational. Independent initial deletions put $1 < \rho = \beta/\alpha < 2$, and a further common deletion gives

$$0 < a_n = \lfloor 2^n \alpha \rfloor < b_n = \lceil 2^n \beta \rceil < a_{n+1} \quad (n \geq 0),$$

with all values above the prescribed finite deletion. No interchange of rounding conventions is needed. The continuation lemma applies, and a band continued through level $n - 1$ has width $W_n = (\alpha + \beta)2^n + O(n)$.

Choose a large convergent denominator q of ρ . Let P be the last convergent denominator at most $q/128$, let Q be its numerator, and let P^+ be the successor denominator. Thus $q/128 < P^+ \leq q$. Applying Lemma 6.1 to $e = \alpha|P\rho - Q|$ gives a unit level n satisfying

$$\frac{P^+}{2\alpha} < 2^n < \frac{4P^+}{\alpha}, \quad q \geq 128P, \quad \frac{\alpha}{4} 2^n < q < 256\alpha 2^n. \quad (6.3)$$

These follow from the same continued-fraction bounds as Lemma 4.1; in particular $P, n \rightarrow \infty$ as $q \rightarrow \infty$.

Let the current gap bound be $g \geq 2$. Continue the band through level $n - 1$ and apply the unit, obtaining a represented $(g - 1)$ -band Y of width $W_n + 1$. Put

$$H = \lceil \log_2(P + Q) \rceil, \quad t = n + H + 1, \quad A = \alpha 2^t, \quad B = \rho A.$$

The restoration uses only levels at least t . For large P , $Q/P < 2$, so

$$2^{H+1} < 4(P + Q) < 12P, \quad \frac{3A}{q} < \frac{9\alpha}{32} 2^n.$$

Choose the least L such that $K = 2^{L+1} - 1 \geq 16q$, and realize each $Au + Bv$, $0 \leq u, v \leq K$, by the binary masks on the fresh rays. The floor and ceiling errors have opposite signs and each has

magnitude at most $L + 1$, hence the combined error has magnitude at most $L + 1$. The rectangle and perturbation lemmas give a family of actual shifts with gaps at most

$$\frac{3A}{q} + 2(L + 1) < W_n$$

for sufficiently large q , since $L = O(n)$. Their translates of Y form a represented $(g - 1)$ -band.

The nominal shift span is at least $A((1 + \rho)K - 4q - 1)$. Subtracting $B(K + 1)$ leaves at least $A(12q - 3)$, which eventually exceeds $2(L + 1) + 2$. As $2b_{t+L} \leq B(K + 1) + 2$, the actual span restores width at least $2b_{t+L}$. The old band, the unit, and the restoration use the disjoint ranges below n , $[n, n + H]$, and $[t, t + L]$. Thus the continuation condition is restored in each round. Finite descent of g proves strong completeness.

6.3. Equal dyadic orbits. Suppose $\alpha/\beta \in 2^{\mathbb{Z}}$ and the coefficients are not dyadic rationals. Initial deletions align both rays at one nondyadic coefficient ξ . Their terms at level n are then

$$a_n = \lfloor \xi 2^n \rfloor, \quad a_n + 1 = \lceil \xi 2^n \rceil.$$

After a further deletion, $a_n \geq 2$ and $a_n < a_n + 1 < a_{n+1}$ at every level.

Lemma 6.2. *Let $g \geq 2$ and let X be a represented g -band of width $W \geq a$. Suppose $\{a, a + 1\}$ and $\{a', a' + 1\}$ are disjoint pairs of fresh terms with $a < a' \leq 2a + 1$. Together with these four terms, X represents a $(g - 1)$ -band of width $W + 2a' + 2$.*

Proof. Write $\ell = \min X$ and form

$$\begin{aligned} B_1 &= (X + a) \cup (X + a + 1), & X' &= X + \{0, a, a + 1, 2a + 1\}, \\ B_2 &= (X' + a') \cup (X' + a' + 1), & B_3 &= B_1 + 2a' + 1. \end{aligned}$$

The shifts defining X' differ successively by $a, 1, a$, all at most $W + 1$, so X' is a represented g -band of width $W + 2a + 1$. The gap-reduction lemma makes B_1 and B_2 represented $(g - 1)$ -bands; B_3 is a translate of B_1 using both terms of the second pair. Their spanning intervals are, respectively,

$$[\ell + a, \ell + a + W + 1], \quad [\ell + a', \ell + W + 2a + a' + 2], \quad [\ell + a + 2a' + 1, \ell + a + W + 2a' + 2].$$

The first two overlap because $a' \leq 2a + 1 \leq a + W + 1$. The second and third overlap by the same inequality. Their union is a $(g - 1)$ -band with the claimed width. Every realization uses subsets of the two pairs without repetition. $\square$

An initial continuation band exists, since prefix mass divided by the largest term tends to 4. Continue it to a late level $n - 1$; its width is $W_n = 2\xi 2^n + O(n) \geq a_n$. Apply Lemma 6.2 to levels n and $n + 1$. The new width is at least $2(a_{n+1} + 1)$, so continuation resumes from level $n + 2$. Finite descent again proves strong completeness.

6.4. Distinct rational dyadic orbits. For the remaining ratios, initial deletions put the rays in the form

$$a_n = \lfloor P 2^n \tau \rfloor, \quad b_n = \lceil Q 2^n \tau \rceil, \tag{6.4}$$

where P, Q are distinct coprime odd positive integers and τ is nondyadic. A common deletion makes $\tau \geq 3$ and places every term above the prescribed finite deletion. The distinct integers $P 2^i, Q 2^j$ differ by at least one, so all rounded values are distinct. Put

$$x = 2^n \tau, \quad \varepsilon_n = a_{n+1} - 2a_n, \quad \eta_n = 2b_n - b_{n+1};$$

both carries lie in $\{0, 1\}$.

Lemma 6.3. *There is an integer H , depending only on P, Q , such that at every level n the intervals $I_{Q,H}^-(Px)$ and $I_{P,H}^+(Qx)$ supply adjacent integers, with one realization the ordinary binary expansion on a ray whose coefficient is at least 3. When $P = 1 < Q$ the ordinary realization can be on the second ray; when $Q = 1 < P$ it can be on the first.*

Proof. Write

$$\theta_n = \lceil Qx \rceil - Qx, \quad \varphi_n = Px - \lfloor Px \rfloor.$$

Then the integer $d_n = Pb_n - Qa_n = P\theta_n + Q\varphi_n$ satisfies $0 < d_n < P + Q$. Positivity follows because equality to zero would make Px, Qx , hence x , integers, contradicting nondyadicity. Also

$$d_{n+1} = 2d_n - P\eta_n - Q\varepsilon_n.$$

With $k_0 = \lceil \log_2(P + Q) \rceil$, every k_0 consecutive levels contain a nonzero carry: otherwise d_n would be multiplied by $2^{k_0} \geq P + Q$.

Let s be the sum of the binary digit counts of P and Q . Choose $H \geq 2sk_0$ with $2^H > \max(P, Q)$. Since both multipliers are odd, the interval widths count the carries over the full range of H levels. At least $2s$ carries occur, so one interval has width at least s . The difference between the ordinary endpoints is the integer

$$\Theta = F_{Qx}^+(P) - F_{Px}^-(Q) = \sum_i \text{bit}_i(P)\theta_{n+i} + \sum_i \text{bit}_i(Q)\varphi_{n+i}.$$

Here $1 \leq \Theta < s$: the lower bound uses the nonzero sum $\theta_n + \varphi_n$ and the oddness of P, Q . If the floor interval has width at least s , it contains $F_{Qx}^+(P) - 1$, adjacent to the ordinary ceiling endpoint. If the ceiling interval has width at least s , it contains $F_{Px}^-(Q) + 1$, adjacent to the ordinary floor endpoint. When $P, Q \geq 3$, either ordinary ray has the required coefficient.

If $P = 1 < Q$, retain the ordinary ceiling term $\lceil Qx \rceil$. The floor interval contains $\lfloor Qx \rfloor$, which is adjacent unless Qx is an integer. In that case a singleton floor interval would force $\{x\} < 2^{-H}$. The integer $Q\{x\}$ would then lie in $[0, 1)$, forcing x to be integral. Hence the floor interval is nondegenerate and contains a neighbour of Qx . For $Q = 1 < P$, use the ordinary floor term and the same argument with $\lfloor x \rfloor - x < 2^{-H}$. $\square$

Lemma 6.4. *Let $G = 2^h$, $0 < K < G$, and let C be odd. Allow z to have binary digits only where $G - 1 - K$ has a one, and w only where K has a one. Then*

$$(z, w) \mapsto Cz + w \pmod{G}$$

is a bijection onto $\mathbb{Z}/G\mathbb{Z}$. In particular, the residues $Cz \pmod{G}$ have cyclic gaps at most $K + 1$.

Proof. For any target residue, determine the input bits in increasing order. At each bit exactly one input is available: the bit of z if the bit of K is zero, and that of w otherwise. Its coefficient is respectively C or 1, both odd; higher input bits do not affect the current bit. Thus every choice is forced and every target is attained uniquely. Since $0 \leq w \leq K$, each target lies at most K steps after a residue Cz around the circle. A run of $K + 1$ missing residues would contradict this for its last residue. The cyclic gap bound follows. $\square$

The width is restored after the canonical unit as follows. An initial band exists because $S_m/H_m \rightarrow 2(P + Q)/\max(P, Q) > 2$, where $H_m = \max(a_m, b_m)$. During any fixed round, continuation to level $n - 1$ gives

$$W_n = (P + Q)x + O(n). \tag{6.5}$$

The unit from [Lemma 6.3](#) produces a $(g - 1)$ -band Y of width $W_n + 1$.

Let C denote the coefficient of its ordinary ray and let K be the other coefficient. Thus $\{C, K\} = \{P, Q\}$, $C \geq 3$, and the ordinary multiplier is K . Let $G = 2^h$ be the least power of two exceeding K . Enlarge the common H so that $H \geq h$ for either choice. The unit leaves unused the zero-digit positions of K below h on the ordinary ray, every position from h onward on that ray, and every position from $H + 1$ onward on the other ray.

Up to level $n + L$, the two unused tails have nominal subset sums

$$Gx(cu + dv), \quad c = C, \quad d = K2^{H+1-h}, \quad 0 \leq u \leq A_L = 2^{L-h+1} - 1, \quad 0 \leq v \leq B_L = 2^{L-H} - 1.$$

The integers c, d are coprime. Put

$$R = \max\{c(d-1), d(c-1)\}, \quad Z_L = cA_L + dB_L.$$

Choose L large enough that $A_L \geq d-1$ and $B_L \geq c-1$. By Lemma 5.2, every integer $j \in [R, Z_L - R]$ has a bounded representation $cu + dv$; fix one corresponding mask for each j . Its actual sum is $Gjx + O_{P,Q,H,L}(1)$.

For a zero-digit mask ζ of K below h , write

$$C\zeta = Gv_\zeta + r_\zeta, \quad 0 \leq r_\zeta < G, \quad V_* = \max_{\zeta} v_\zeta.$$

The actual zero-digit sum is $C\zeta x + O(1)$. For every $R + V_* \leq j \leq Z_L - R$, combine this sum with the tail mask of index $j - v_\zeta$. That index lies in $[R, Z_L - R]$, and the two masks use disjoint terms. Their nominal sum is $(Gj + r_\zeta)x$. Keeping one mask for each residue, Lemma 6.4 shows that the consecutive nominal gaps are at most $(K+1)x$, including the gaps between successive periods. Since

$$K+1 < C + K = P + Q,$$

the actual translates of Y overlap for sufficiently large n , by (6.5). Their union is a represented $(g-1)$ -band Y' .

Because residue zero occurs, the span of the actual shifts is at least

$$G(Z_L - 2R - V_*)x - O_{P,Q,H,L}(1).$$

Moreover

$$GZ_L = (P + Q)2^{L+1} - CG - K2^{H+1}.$$

Choose L first, sufficiently large that

$$G(Z_L - 2R - V_*) > 2 \max(P, Q)2^L.$$

This is possible because the difference of the leading terms is $2 \min(P, Q)2^L$. One L can meet all conditions for either possible ordinary ray. Only then choose n sufficiently large to absorb the rounding errors and the $O(n)$ term in (6.5). The width of Y' then exceeds $2H_{n+L}$, restoring continuation. Within each representation the old band, the unit, the two tails, and the zero-digit mask use disjoint terms. Finite descent completes this case.

Proof of Theorem B. Necessity is Proposition 2.5(a). Otherwise the ratio is irrational, a power of two, or rational but not a power of two. The preceding three arguments give completeness after every prescribed finite deletion in these respective cases. Therefore $E_{\alpha,\beta}^\pm$ is strongly complete exactly when the coefficients are not both dyadic rationals. $\square$

7. DISJOINT SUPPLIES

The unit that lowers the gap bound and the rectangle that restores the width are chosen together in each round. The following proposition shows that they cannot be drawn from two supplies fixed in advance: no two disjoint subsets of the two floor rays both have unbounded subset sums with bounded gaps. An unbounded set of nonnegative integers has *bounded gaps* if the differences between consecutive elements are bounded; the set of subset sums of a complete set has bounded gaps, and the converse fails, for instance for the even numbers.

Lemma 7.1 (Carry budget). *Let E be an infinite set of positive integers such that $\text{FS}(E)$ has gaps at most g . For a real $x > 0$ let T be the sum of the elements of E that are less than x , and let y be the least element of E that is at least x . Then $y - T \leq g$.*

Proof. If $y \leq T$ there is nothing to prove. Otherwise T and y are elements of $\text{FS}(E)$ and no element of $\text{FS}(E)$ lies strictly between them: a sum that uses an element of E at least x is at least y , and a sum that uses none is at most T . Hence $y - T \leq g$. $\square$

Proposition 7.2. *Let $0 < \alpha < \beta < 2\alpha$, suppose that the floor sequences $a_n = \lfloor 2^n \alpha \rfloor$ and $b_n = \lfloor 2^n \beta \rfloor$ satisfy (2.5), and suppose that at least one coefficient is not a dyadic rational. There are no disjoint subsets E_1, E_2 of $\{a_n, b_n : n \geq 0\}$ such that both $\text{FS}(E_1)$ and $\text{FS}(E_2)$ are unbounded with bounded gaps.*

Proof. Write $\varepsilon_j = a_{j+1} - 2a_j$ and $\eta_j = b_{j+1} - 2b_j$, both in $\{0, 1\}$, and $S_{n-1} = \sum_{j < n} (a_j + b_j)$ as in (2.1). Summing the recurrences gives

$$a_n + b_n - S_{n-1} = a_0 + b_0 + \sum_{j < n} (\varepsilon_j + \eta_j), \quad (7.1)$$

and the right-hand side tends to infinity, because a coefficient that is not a dyadic rational has infinitely many nonzero carries (Section 3).

Suppose that E_1, E_2 exist, with gap bounds g_1, g_2 ; both are infinite. Apply Lemma 7.1 with $x = a_n$: let T_i be the sum of the elements of E_i below a_n and x_i the least element of E_i that is at least a_n . By (2.5) the terms that are at least a_n are those of index at least n , the two smallest being $a_n < b_n$; since $x_1 \neq x_2$,

$$x_1 + x_2 \geq a_n + b_n, \quad T_1 + T_2 \leq S_{n-1},$$

the second because E_1, E_2 are disjoint. Hence

$$g_1 + g_2 \geq (x_1 - T_1) + (x_2 - T_2) \geq a_n + b_n - S_{n-1},$$

which is unbounded in n by (7.1), a contradiction. $\square$

The corresponding statement for ceilings is false. If the terms $u_j = \lceil \xi 2^j \rceil$ are distinct and positive, write $u_{j+1} = 2u_j - \varepsilon_j$ with $\varepsilon_j \in \{0, 1\}$. Then

$$u_n - \sum_{j < n} u_j = u_0 - \sum_{j < n} \varepsilon_j \leq u_0,$$

so, starting from $\{0, u_0\}$, Lemma 2.2 shows inductively that all finite-prefix subset-sum sets have gaps at most u_0 , and so has $\text{FS}(\{u_j : j \geq 0\})$. Two interleaving ceiling rays thus give two disjoint bounded-gap supplies; this observation alone does not establish completeness of either ray.

8. BASES BETWEEN ONE AND TWO

Throughout, γ is the positive root of

$$P(x) = x^4 - x^3 - x^2 - x - 1, \quad (8.1)$$

and for a root u of P we put

$$c_u = \frac{1}{(u-1)P'(u)} = \frac{1}{u^3(5u-8)}, \quad c = c_\gamma. \quad (8.2)$$

The second equality holds because $Q(x) := (x-1)P(x) = x^5 - 2x^4 + 1$ has $Q'(x) = x^3(5x-8)$ and $Q'(u) = (u-1)P'(u)$ at a root u of P .

8.1. The tetranacci pair.

Lemma 8.1. *The roots of P are γ , a real number r , and a nonreal pair $z, \bar{z}$, all simple, with*

$$\frac{19}{10} < \gamma < \frac{39}{20}, \quad -\frac{4}{5} < r < -\frac{19}{25}, \quad \frac{4}{5} < |z| < \frac{5}{6}, \quad \text{Re } z < -\frac{1}{20}.$$

Proof. By Descartes's rule P has exactly one positive root, and $P(19/10) = -3369/10000 < 0 < 46661/160000 = P(39/20)$ locates it. For $t > 0$, $P'(-t) = -4t^3 - 3(t-1/3)^2 - 2/3 < 0$, so P has exactly one negative root, located by $P(-4/5) = 51/625 > 0 > -17579/390625 = P(-19/25)$; the remaining two roots are a nonreal conjugate pair. From the product and the sum of the roots, $|z|^2 = 1/(\gamma|r|)$ and $\text{Re } z = (1 - \gamma - r)/2 < -1/20$, and the bounds on γ and r give

$16/25 < 25/39 < |z|^2 < 250/361 < 25/36$. Since neither 0 nor $8/5$ is a root of Q , the polynomial Q , hence P , has only simple roots. $\square$

Lemma 8.2. *For $m \geq 0$ put $\Sigma_m = \sum_{P(u)=0} c_u u^m$. Then $\Sigma_m \in \frac{1}{3} + \mathbb{Z}$, the differences $\Delta_m = \Sigma_{m+1} - \Sigma_m$ are the integers determined by $\Delta_0 = \Delta_1 = \Delta_2 = 0$, $\Delta_3 = 1$ and $\Delta_{m+4} = \Delta_{m+3} + \Delta_{m+2} + \Delta_{m+1} + \Delta_m$, and*

$$|\Sigma_m - c\gamma^m| < \frac{7}{10} \left(\frac{5}{6}\right)^m. \quad (8.3)$$

Proof. The partial fractions $1/P(x) = \sum_{P(u)=0} 1/(P'(u)(x-u))$ at $x = 1$ give $\Sigma_0 = -1/P(1) = 1/3$. Next, $\Delta_m = \sum_u c_u(u-1)u^m = \sum_u u^m/P'(u)$, which is 0 for $0 \leq m \leq 2$ and 1 for $m = 3$ by comparing the coefficients of x^3 in the Lagrange interpolation of $1, x, x^2, x^3$ at the four roots; the recurrence holds because every root satisfies (8.1). Hence the Δ_m are integers and $\Sigma_m = \frac{1}{3} + \sum_{j < m} \Delta_j$. By Lemma 8.1, $|c_r| < 1/((19/25)^3(8 + 19/5)) < 1/5$, and $|5z - 8| \geq 8 - 5\operatorname{Re} z > 8$ gives $|c_z| = |c_{\bar{z}}| < 1/((4/5)^3 8) < 1/4$; since $|r|, |z| < 5/6$, the three terms other than $c\gamma^m$ contribute less than $(1/5 + 1/4 + 1/4)(5/6)^m$. $\square$

Proposition 8.3. *Put $s_m = \sum_{j < m} \Delta_j = \Sigma_m - \frac{1}{3} \in \mathbb{Z}$. Then $\lfloor 2c\gamma^m \rfloor = 2s_m$ for every $m \geq 4$, and $(s_4, s_5, s_6, s_7) = (1, 2, 4, 8)$.*

Proof. For $m \geq 8$, (8.3) gives $|\Sigma_m - c\gamma^m| < \frac{7}{10}(5/6)^8 = 546875/3359232 < 1/6$, so

$$s_m + \frac{1}{6} < c\gamma^m < s_m + \frac{1}{2}, \quad \text{hence} \quad \lfloor 2c\gamma^m \rfloor = 2s_m \quad (m \geq 8). \quad (8.4)$$

For $4 \leq m \leq 7$ put $\ell = 771/400$ and $v = 4819/2500$; then

$$P(\ell) = -\frac{20061919}{25600000000} < 0 < \frac{18784065621}{39062500000000} = P(v),$$

so $\ell < \gamma < v$. For $1 \leq j \leq 4$ the function $f_j(x) = 2x^j/(5x - 8)$ satisfies $f_j(\gamma) = 2c\gamma^{j+3}$ by (8.2) and is strictly decreasing on $[\ell, v]$, its derivative having the sign of $5(j-1)x - 8j < 0$ there; so $f_j(v) < 2c\gamma^{j+3} < f_j(\ell)$, and the exact values

j	$f_j(v) - 2^j$	$2^j + 1 - f_j(\ell)$
1	1448/4095	423/655
2	2747761/5118750	60559/131000
3	9535485259/12796875000	13285989/52400000
4	27421628463121/31992187500000	2959897519/20960000000

are positive, so $(\lfloor 2c\gamma^4 \rfloor, \dots, \lfloor 2c\gamma^7 \rfloor) = (2, 4, 8, 16)$. The recurrence for Δ_m gives $(s_4, \dots, s_7) = (1, 2, 4, 8)$, which agrees. $\square$

Proof of Theorem C(ii). By (8.2), $\alpha = 2c\gamma^4$ and $\beta = 2c\gamma^5$, so $\lfloor \alpha\gamma^n \rfloor = 2s_{n+4}$ and $\lfloor \beta\gamma^n \rfloor = 2s_{n+5}$ for $n \geq 0$ by Proposition 8.3; all terms are even. The monic polynomial P has no rational root, since $P(\pm 1) \neq 0$, so γ and $\alpha/\beta = 1/\gamma$ are irrational, and $\gamma > 19/10 > 8/5$ gives $\alpha, \beta > 0$.

Every finite sum of terms is even. Conversely, the Δ_j are nonnegative with $\Delta_j \geq 1$ for $j \geq 3$, and for $m \geq 4$, $\Delta_m = \Delta_{m-1} + \dots + \Delta_{m-4} \leq s_m$, so $s_{m+1} \leq 2s_m$. With $a_j = s_{j+4}$ this gives $a_0 = 1$ and $a_{j+1} \leq 2a_j$, whence $a_j \leq 1 + \sum_{i < j} a_i$ by induction; so if the subset sums of $a_0, \dots, a_{j-1}$ fill $[0, \sum_{i < j} a_i]$ then those of $a_0, \dots, a_j$ fill $[0, \sum_{i \leq j} a_i]$. As the sums tend to infinity, every nonnegative integer is a sum of distinct terms a_j , and doubling shows that every nonnegative even integer is a sum of distinct terms $\lfloor \alpha\gamma^n \rfloor$ of the first sequence alone. $\square$

8.2. An off-orbit pair. The second sequence in [Theorem C\(ii\)](#) is a shift of the first. The obstruction persists at the same base when the coefficient ratio lies outside $\mathbb{Q}\gamma^{\mathbb{Z}}$.

Corollary 8.4. *Put $\alpha = 2c\gamma^{16}$ and $\beta = (1 + 3\gamma)\alpha$. Then $\lfloor \alpha\gamma^n \rfloor$ and $\lfloor \beta\gamma^n \rfloor$ are even for every $n \geq 0$, and $\beta/\alpha \notin \mathbb{Q}\gamma^j$ for every $j \in \mathbb{Z}$.*

Proof. Write $E_k = \sum_{u \neq \gamma} c_u u^k$, so that $c\gamma^k = s_k + \frac{1}{3} - E_k$ with s_k as in [Proposition 8.3](#), and $|E_k| < \frac{7}{10}(5/6)^k$ by the proof of [Lemma 8.2](#). Then $|E_k + 3E_{k+1}| < \frac{49}{20}(5/6)^k < \frac{1}{6}$ for $k \geq 16$, the last step being $147 \cdot 5^{16} < 10 \cdot 6^{16}$, and the same bound holds for $|E_k|$. Hence, for $k \geq 16$,

$$2c\gamma^k = 2s_k + \frac{2}{3} - 2E_k, \quad 2c\gamma^k(1 + 3\gamma) = 2(s_k + 3s_{k+1} + 1) + \frac{2}{3} - 2(E_k + 3E_{k+1}),$$

and in both cases the fractional part lies in $(\frac{1}{3}, 1)$, so the integer parts are $2s_k$ and $2(s_k + 3s_{k+1} + 1)$. This proves the parity assertions with $k = 16 + n$.

Modulo 2, P becomes $x^4 + x^3 + x^2 + x + 1$, which has no root in $\mathbb{F}_2$ and is not divisible by $x^2 + x + 1$; so P is irreducible over $\mathbb{Q}$ and $\mathbb{Q}(\gamma)$ has degree 4. Its norms satisfy $N(\gamma) = P(0) \cdot (-1)^4 = -1$ and $N(1 + 3\gamma) = 3^4 P(-1/3) = -59$. If $1 + 3\gamma = u\gamma^j$ with $u \in \mathbb{Q}$, then $u^4 = \pm 59$, which is impossible because 59 is prime. $\square$

8.3. Divisibility obstructions at bases tending to two. For an integer $m \geq 4$ let $P_m(x) = x^m - x^{m-1} - \dots - x - 1$. Since $\sum_{j=1}^m x^{-j}$ decreases for $x > 0$ and takes the values $m > 1$ at $x = 1$ and $1 - 2^{-m} < 1$ at $x = 2$, P_m has a unique positive root $\gamma_m \in (1, 2)$; and $(x-1)P_m(x) = x^{m+1} - 2x^m + 1$ gives $2 - \gamma_m = \gamma_m^{-m} < (2/3)^m$, using $\gamma_m > 3/2$ (the first two terms of $\sum_{j \leq m} (3/2)^{-j}$ already exceed 1). So $\gamma_m \rightarrow 2$.

Lemma 8.5. *All roots of P_m are simple, and every root $u \neq \gamma_m$ satisfies $|u| \leq \gamma_m - 1 < 1$.*

Proof. Write $\gamma = \gamma_m$ and $P_m(x) = (x-\gamma)(x^{m-1} + a_1x^{m-2} + \dots + a_{m-1})$, so that $a_0 = 1$, $a_k = \gamma a_{k-1} - 1$ for $1 \leq k \leq m-1$, and $a_{m-1} = 1/\gamma$ from the constant term. The backward recurrence $a_{k-1} = (a_k + 1)/\gamma$ shows that all a_k are positive; then, by induction from $a_0 = 1$, $a_k - a_{k-1} = (\gamma - 1)a_{k-1} - 1 < 0$, so $a_k \leq 1$, and $a_k/a_{k-1} = \gamma - 1/a_{k-1} \leq \gamma - 1$. Put $\rho = \gamma - 1$ and $b_k = a_k/\rho^k$, so that $1 = b_0 \geq b_1 \geq \dots \geq b_{m-1} > 0$, and let $R(w) = w^{m-1} + b_1w^{m-2} + \dots + b_{m-1}$. If $R(w) = 0$ and $|w| > 1$, then multiplying by $w-1$ and rearranging gives $|w|^m \leq \sum_{k=1}^{m-1} (b_{k-1} - b_k)|w|^{m-k} + b_{m-1} \leq |w|^{m-1}$, a contradiction. So every root of the quotient $x^{m-1} + a_1x^{m-2} + \dots + a_{m-1} = \rho^{m-1}R(x/\rho)$ has modulus at most ρ . For simplicity, $G(x) := (x-1)P_m(x) = x^{m+1} - 2x^m + 1$ has $G'(x) = x^{m-1}((m+1)x - 2m)$; $G(0) = 1$, and at $t = 2m/(m+1)$, where $x^m(2-x)$ attains its maximum on $(0, 2)$ and exceeds its value 1 at $x = 1$, $G(t) = 1 - t^m(2-t) < 0$. So G and P_m have no repeated root. $\square$

Put

$$c_m = \frac{1}{(\gamma_m - 1)P'_m(\gamma_m)}, \quad (8.5)$$

which is positive because $P_m(x) = x^m(1 - \sum_{j \leq m} x^{-j})$ has positive derivative at γ_m ; for $m = 4$ this is the constant c of [\(8.2\)](#).

Proof of Theorem C(iii). Let $u_n = \sum_{P_m(r)=0} r^n / P'_m(r)$. By Lagrange interpolation $u_n = 0$ for $0 \leq n < m-1$ and $u_{m-1} = 1$, and each r^n satisfies the recurrence $u_{n+m} = u_{n+m-1} + \dots + u_n$, so all u_n are integers. Put $s_n = \sum_{j < n} u_j$ and $c_r = 1/((r-1)P'_m(r))$; then $c_r r^{n+1} - c_r r^n = r^n / P'_m(r)$, and $\sum_r c_r = -1/P_m(1) = 1/(m-1)$, so

$$c_m \gamma_m^n = s_n + \frac{1}{m-1} - e_n, \quad e_n = \sum_{r \neq \gamma_m} c_r r^n \in \mathbb{R}, \quad (8.6)$$

and by [Lemma 8.5](#), $|e_n| \leq C_m \rho_m^n$ with $\rho_m = \gamma_m - 1$ and $C_m = \sum_{r \neq \gamma_m} |c_r|$. Choose N with

$$C_m \rho_m^N < \min \left\{ \frac{1}{m-1}, \frac{1}{d} - \frac{1}{m-1} \right\},$$

which is possible since $0 < \rho_m < 1$ and $d < m - 1$. For $k \geq N$, (8.6) gives $s_k < c_m \gamma_m^k < s_k + 1/d$, hence $\lfloor dc_m \gamma_m^k \rfloor = ds_k$; so $\lfloor \alpha \gamma_m^n \rfloor = ds_{N+n}$ and $\lfloor \beta \gamma_m^n \rfloor = ds_{N+n+1}$. A rational root of the monic polynomial P_m would be an integer, and there is none in $(1, 2)$; so γ_m and $\beta/\alpha = \gamma_m$ are irrational. Every finite subset sum is divisible by d . $\square$

Corollary 8.6. *Let $m \geq 4$, $2 \leq d < m - 1$, and let $f \in \mathbb{Z}[x]$ have nonnegative coefficients and $f(1) \equiv 1 \pmod{m-1}$. There is $N \geq 0$ such that, with $\alpha = dc_m \gamma_m^N$ and $\beta = f(\gamma_m)\alpha$, every term $\lfloor \alpha \gamma_m^n \rfloor$ and $\lfloor \beta \gamma_m^n \rfloor$ is divisible by d . For $f(x) = x + m - 1$ the ratio $\beta/\alpha = \gamma_m + m - 1$ is irrational.*

Proof. Write $f(x) = \sum_j f_j x^j$ and $f(1) = 1 + (m-1)t$. By (8.6),

$$c_m \gamma_m^n f(\gamma_m) = \sum_j f_j \left(s_{n+j} + \frac{1}{m-1} - e_{n+j} \right) = \left(\sum_j f_j s_{n+j} + t \right) + \frac{1}{m-1} - \sum_j f_j e_{n+j},$$

with $|\sum_j f_j e_{n+j}| \leq C_m \rho_m^n \sum_j f_j \rho_m^j$. Choose N so that

$$C_m \rho_m^N \max \left\{ 1, \sum_j f_j \rho_m^j \right\} < \min \left\{ \frac{1}{m-1}, \frac{1}{d} - \frac{1}{m-1} \right\};$$

then, as in the proof of Theorem C(iii), $\lfloor dc_m \gamma_m^k f(\gamma_m) \rfloor = d(\sum_j f_j s_{k+j} + t)$ for $k \geq N$, and the same choice of N serves the first sequence. Since γ_m is irrational, so is $\gamma_m + m - 1$. $\square$

8.4. Roots of two.

Proof of Theorem C(i). Put $\theta = 2^{1/k}$. The number θ is irrational: a rational root of the monic polynomial $x^k - 2$ would be an integer, whereas $1 < \theta < 2$. The terms $\lfloor t\theta^n \rfloor$ with $n \equiv 0$ and $n \equiv 1 \pmod{k}$ form, respectively, the two dyadic rays with coefficients t and $t\theta$, whose ratio is irrational; the same holds for ceilings. By Theorem 4.4 the set of positive floor values of these two rays, and the set of their ceiling values, are strongly complete, and each is contained in the corresponding single ray at the base θ . A set containing a strongly complete set is strongly complete, since the deletion of finitely many elements removes only finitely many elements of the contained set. The assertion about two rays follows by the same inclusion. $\square$

APPENDIX A. EXACT COMPUTATIONS

A.1. A pair at base two. Take

$$\alpha = 2\sqrt{2}, \quad \beta = 2 + 2\sqrt{2}, \quad a_n = \lfloor 2^n \alpha \rfloor, \quad b_n = \lfloor 2^n \beta \rfloor = a_n + 2^{n+1}.$$

The ratio is $1 + 1/\sqrt{2}$, so Theorem A gives strong completeness. For the undeleted set the exact completeness threshold follows from a short represented interval.

Proposition A.1. *For this pair, every integer $N \geq 13$ belongs to $\text{FS}(E_{\alpha,\beta}^-)$, and 12 does not.*

Proof. The first four pairs are

$$(a_n, b_n) = (2, 4), (5, 9), (11, 19), (22, 38).$$

The first six terms have total 50 and represent each integer from 13 to 25:

13	9 + 4	20	11 + 9
14	9 + 5	21	19 + 2
15	9 + 4 + 2	22	11 + 9 + 2
16	11 + 5	23	19 + 4
17	11 + 4 + 2	24	19 + 5
18	11 + 5 + 2	25	19 + 4 + 2
19	19		

Complementing these subsets represents [25, 37]. Adjoining 22 joins [13, 37] to [35, 59], and adjoining 38 joins [13, 59] to [51, 97]. Hence

$$[13, 97] \subseteq \text{FS}\{a_j, b_j : 0 \leq j \leq 3\}, \quad 97 - 13 = 84 \geq 76 = 2b_3.$$

Continuation represents every integer at least 13. A representation of 12 could use only 2, 4, 5, 9, 11. Including 11 would require a further 1; including 9 but not 11 would require a further 3; excluding both leaves total at most $2 + 4 + 5 = 11$. Thus 12 is not represented. $\square$

One round of the unit-and-restoration construction can be followed exactly for this pair. Let X contain all subset sums of the eight terms through level 3. Its 2^8 masks give 101 distinct sums between 0 and 110, with largest gap 2. The disjoint blocks

$$b_4 + b_8 = 77 + 1236 = 1313, \quad a_4 + a_6 + a_7 + a_8 = 45 + 181 + 362 + 724 = 1312$$

give

$$(X + 1312) \cup (X + 1313) = [1312, 1423].$$

The old band uses levels 0–3, and the unit blocks use levels 4–8. Take the subsequent supply from levels 10 onward. Exact unions with its subset sums contain the following intervals:

last level	represented interval	width	$2b_m$
17	[196 271, 1 805 812]	1 609 541	1 265 742
26	[196 271, 1 027 484 186]	1 027 287 915	648 060 518

Thus restoration is already complete at level 17. The accompanying script `verify_354.py` reproduces both computations using only integer arithmetic, with

$$a_n = \text{isqrt}(2^{2n+3}).$$

It stores subset-sum sets as exact unions of integer intervals and can reconstruct distinct-term representations, both from the complete tail beginning at 13 and from the displayed finite restoration. These computations illustrate the construction; the universal statement follows from the preceding proofs.

A.2. The tetranacci pair. For the pair in [Theorem C\(ii\)](#), the integer recurrence of [Proposition 8.3](#) begins

$$(s_4, \dots, s_{12}) = (1, 2, 4, 8, 16, 31, 60, 116, 224).$$

The proved identity $\lfloor \alpha \gamma^n \rfloor = 2s_{n+4}$ therefore gives the initial terms

$$2, 4, 8, 16, 32, 62, 120, 232, 448.$$

The accompanying script `check_gamma.py` uses only integers and rational numbers. It verifies the root-enclosure signs, coefficient and tail bounds, initial intervals, recurrence identities, and norm obstruction used in [Lemmas 8.1](#) and [8.2](#), [Proposition 8.3](#), and [Corollary 8.4](#). Independent rational isolation of the positive root also checks the first eight terms of the displayed ray and the first six pairs in [Corollary 8.4](#). The all-index conclusions use the conjugate estimates and recurrence arguments in the text, rather than extrapolation from these finite checks.

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LONDON, UNITED KINGDOM

Email address: `budden@pingyou.com`