

COMPLETE EXPONENTIAL SEQUENCES AT ROOTS OF TWO: ERDŐS PROBLEM #349

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ABSTRACT. We prove that for every $k \geq 2$ and every $t > 0$ the sequence $\lfloor t2^{n/k} \rfloor$, and its ceiling counterpart, is strongly complete: after any finite deletion, every sufficiently large integer is a sum of distinct terms. Exactly two residue classes of exponents modulo k are needed; a single class is never strongly complete, and its floor values are complete only for the powers of two. No k pairwise disjoint subsets of the sequence have subset sums with bounded gaps, while $\lfloor k/2 \rfloor$ pairwise disjoint strongly complete subsets exist; strong completeness at any base forces the base to lie below the golden ratio.

We show that these statements follow from the strong completeness of two dyadic sequences $\lfloor 2^n \alpha \rfloor$, $\lfloor 2^n \beta \rfloor$ with irrational coefficient ratio, for floors and for ceilings, which we prove: signed binary carries lower the gaps of a represented set, and continued fractions restore its width from unused terms, with no hypothesis on partial quotients or on the frequency of carries. Exact completeness thresholds are computed at the first roots of two. Taken together, one geometric sequence below the golden ratio is complete for every coefficient because it contains two dyadic sequences that cannot both be lattices.

1. INTRODUCTION

For a set E of positive integers, write

$$\text{FS}(E) = \left\{ \sum_{a \in F} a : F \subseteq E \text{ finite} \right\},$$

including the empty sum. The set E is *complete* if $\text{FS}(E)$ contains every sufficiently large integer, and *strongly complete* if $E \setminus F$ is complete for every finite $F \subseteq E$. For $t, \alpha > 0$ put

$$E(t, \alpha) = \{ \lfloor t\alpha^n \rfloor : n \geq 0 \} \setminus \{0\}. \quad (1.1)$$

Erdős and Graham [5, p. 57], following Graham [6, 7], ask for which $t, \alpha > 0$ the sequence $\lfloor t\alpha^n \rfloor$ is complete, that is, for which pairs every sufficiently large integer is a sum of distinct integers of the form $\lfloor t\alpha^n \rfloor$; the question is recorded as Problem 349 at <https://www.erdosproblems.com/349>. An integer attained at several indices is thus available once, which is the set $E(t, \alpha)$; the convention $n \geq 1$ is recovered by replacing t with $t\alpha$. For $k \geq 2$ write $\theta_k = 2^{1/k}$, and for $R \subseteq \{0, \dots, k-1\}$ put

$$E_{k,t}^-(R) = \{ \lfloor t\theta_k^n \rfloor : n \geq 0, n \bmod k \in R \} \setminus \{0\}, \quad (1.2)$$

$$E_{k,t}^+(R) = \{ \lceil t\theta_k^n \rceil : n \geq 0, n \bmod k \in R \}, \quad (1.3)$$

and write $E_{k,t}^\pm$ when all residue classes are retained, so that $E_{k,t}^- = E(t, \theta_k)$.

Theorem A (Roots of two). *For every $k \geq 2$ and every $t > 0$, the sets $E_{k,t}^-$ and $E_{k,t}^+$ are strongly complete. In particular, for any integers $n_0, B_0 \geq 0$, every sufficiently large integer is a sum of distinct values $\lfloor t2^{n/k} \rfloor$ with $n \geq n_0$ and $\lfloor t2^{n/k} \rfloor > B_0$, and likewise for ceilings.*

Theorem B (Exponent classes). *Let $k \geq 2$, $t > 0$ and $R \subseteq \{0, \dots, k-1\}$.*

- (a) *Both $E_{k,t}^-(R)$ and $E_{k,t}^+(R)$ are strongly complete if and only if $|R| \geq 2$.*
- (b) *The floor set $E_{k,t}^-(R)$ is complete if and only if $|R| \geq 2$, or $R = \{r\}$ and $t2^{r/k} = 2^{-m}$ for an integer $m \geq 0$.*

Each residue class is a dyadic sequence with coefficient $t2^{r/k}$, and two different classes have irrational coefficient ratio. The next result separates strong completeness from the possibility of dividing the terms into several independent complete supplies. An unbounded set of nonnegative integers has *bounded gaps* if the differences between consecutive elements are bounded.

Theorem C (Disjoint subsets). *The set $E_{k,t}^-$ contains no k pairwise disjoint subsets whose sets of subset sums are unbounded with bounded gaps. Both $E_{k,t}^-$ and $E_{k,t}^+$ contain $\lfloor k/2 \rfloor$ pairwise disjoint strongly complete subsets, and $E_{k,t}^+$ contains k pairwise disjoint infinite subsets whose subset sums have bounded gaps. If $E(t, \alpha)$ is strongly complete for some $t > 0$, then $1 < \alpha < (1 + \sqrt{5})/2$.*

A *dyadic rational* is a number $m/2^h$ with $m \in \mathbb{Z}$ and $h \geq 0$. The three theorems rest on the following statement about two dyadic sequences.

Theorem D (Two dyadic sequences with irrational ratio). *If $\alpha, \beta > 0$ and α/β is irrational, then each of*

$$\{\lfloor 2^j \alpha \rfloor, \lfloor 2^j \beta \rfloor : j \geq 0\} \setminus \{0\} \quad \text{and} \quad \{\lceil 2^j \alpha \rceil, \lceil 2^j \beta \rceil : j \geq 0\}$$

is strongly complete.

The proof of [Theorem D](#) keeps a finite set of represented integers with consecutive gaps at most g , a *represented g -band*, whose width is at least twice the largest term used. The binary recurrence lets every later term be adjoined without increasing the gap bound, so a represented 1-band of this kind proves completeness. To lower g , signed binary expansions of the two rays produce two disjoint blocks of terms whose sums differ by one; translating the band by both sums reduces the gap bound by one but not the width, and a rectangle of subset sums of later, unused terms, taken at a continued-fraction scale chosen together with the blocks, restores the width. No bound on partial quotients and no frequency estimate for carries is used. [Theorem A](#) then follows because two residue classes of exponents form two dyadic rays with irrational ratio, the necessity in [Theorem B](#) because a single rounded dyadic sequence leaves a gap after every term, and the first assertion of [Theorem C](#) because the carries of $k - 1$ nondyadic coefficients outrun any k gap bounds.

Sections 2 and 3 set up bands and carry intervals, Section 4 proves [Theorem D](#), Section 5 proves [Theorems A](#) and [B](#), Section 6 proves [Theorem C](#), and Section 7 isolates the finite verification behind the exact thresholds of Appendix A.

Graham [6] determined the complete pairs with $0 < t < 1$ and $1 < \alpha < 2$ and showed that, for each j , some coefficient makes the set of complete bases a union of at least j disjoint segments; van Doorn (arXiv:2602.23394, February 2026) settles the range $\alpha \geq (1 + \sqrt{5})/2$ by Graham's method, determines when every positive integer is represented for $1 < \alpha < (1 + \sqrt{5})/2$, and verifies completeness by computer on bounded regions such as $\alpha \leq 1.4$, $t \leq 3$. Geneson (arXiv:2609.25107, 20 September 2026) exhibits, from a theorem of Dubickas [4], a Salem number below the golden ratio at which completeness fails for some t , so the completeness expected for every base below the golden ratio does not hold in general. Hegyvári [8] proved that two dyadic sequences are complete when one coefficient is a dyadic rational and the other is not, and [9] that, for a sequence with $s_{n+1} < \gamma s_n$ for large n and some $\gamma \in (1, 2)$, the set of α for which $\lfloor \alpha s_n \rfloor$ is complete contains with each of its points an interval to the right, under a gap condition removed by Chen and Fang [3]; Ma and Chen [10] showed that this set is never empty, and Bergelson and Simmons [1] study strong completeness for broader classes of sets. To our knowledge, [Theorem A](#) gives the first bases in $(1, (1 + \sqrt{5})/2)$ at which completeness holds for every coefficient. An unrefereed claim posted on the page of the dyadic question (Yu and Chen, github.com/Andrewyzzz/erdos354, 13 September 2026, AI-assisted) concerns the floor case of [Theorem D](#) by a different route; [Theorem D](#) is proved here in full, for floors and ceilings.

2. REPRESENTED BANDS

Indices are nonnegative integers, an integer interval $[u, v]$ means $[u, v] \cap \mathbb{Z}$, and the width of a finite nonempty set is its maximum minus its minimum. We use $\{x\} = x - \lfloor x \rfloor$, $v_2(K)$ for the 2-adic valuation of K , and $\text{bit}_i(K)$ for its binary digit at 2^i . Implied constants may depend on the coefficients and on the band at the beginning of a round, but not on the scale chosen within that round.

Definition 2.1. Let $\mathcal{E}$ be a finite set of positive integers, called the *available terms*, and let $g \geq 1$ be an integer. A *represented g -band* on $\mathcal{E}$ is a nonempty finite set $X \subseteq \text{FS}(\mathcal{E})$ whose consecutive elements differ by at most g . Different elements of X may use different subsets of $\mathcal{E}$. Its *spanning interval* is $[\min X, \max X]$, and its *width* is

$$\text{width}(X) = \max X - \min X.$$

Lemma 2.2 (Union). *Let $X_1, \dots, X_r$ be represented g -bands on the same set of available terms, ordered so that $\min X_1 \leq \dots \leq \min X_r$. If*

$$\min X_{i+1} \leq \max(X_1 \cup \dots \cup X_i) + g \quad (1 \leq i < r),$$

then $X_1 \cup \dots \cup X_r$ is a represented g -band.

Proof. For a finite nonempty set $X \subset \mathbb{R}$, consecutive gaps are at most g precisely when the real intervals $[x, x + g]$, $x \in X$, have union $[\min X, \max X + g]$. The displayed inequalities ensure that the corresponding intervals for $X_1, \dots, X_r$ join successively into one interval. It follows that $X_1 \cup \dots \cup X_r$ has gaps at most g , and each of its elements retains its original representation. $\square$

2.1. Two sequences. Let $\alpha, \beta > 0$. Choose floors or ceilings for each sequence and put

$$a_j = \lfloor 2^j \alpha \rfloor \quad \text{or} \quad a_j = \lceil 2^j \alpha \rceil, \quad b_j = \lfloor 2^j \beta \rfloor \quad \text{or} \quad b_j = \lceil 2^j \beta \rceil,$$

with the chosen convention fixed along each sequence. Assume that all terms are positive and pairwise distinct as integers. Write

$$S_m = \sum_{j=0}^m (a_j + b_j), \quad H_m = \max(a_m, b_m). \quad (2.1)$$

Either rounding convention gives

$$2a_j - 1 \leq a_{j+1} \leq 2a_j + 1, \quad 2b_j - 1 \leq b_{j+1} \leq 2b_j + 1. \quad (2.2)$$

Lemma 2.3 (Continuation). *Let X be a represented g -band on the terms through level m , with width $W \geq 2H_m$. Adjoin the later terms one at a time, replacing the current set Y by $Y \cup (Y + w)$ when the term w is adjoined. After level $k \geq m$, the result is a represented g -band with minimum $\min X$ and width $S_k - \delta$, where $\delta = S_m - W \geq 0$. Thus, immediately before level $n > m$ is adjoined, its width is*

$$W_n = S_{n-1} - \delta = (\alpha + \beta)2^n + O(n). \quad (2.3)$$

An initial band satisfying $W \geq 2H_m$ exists for some m and some finite g .

Proof. Suppose Y has width V , and let w be a positive term unused in its representations. If $w \leq V + 1$, then

$$\min(Y + w) = \min Y + w \leq \max Y + 1 \leq \max Y + g.$$

The union lemma shows that $Y \cup (Y + w)$ is a represented g -band. Its minimum is unchanged and its width is $V + w$.

Now put $H = H_m$. Both terms of level $m+1$ are at most $2H + 1 \leq W + 1$, so they can be adjoined successively. If $l_{m+1} = \min(a_{m+1}, b_{m+1})$, then $l_{m+1} \geq 1$ and $H_{m+1} \leq 2H + 1$. Consequently the new width satisfies

$$W + a_{m+1} + b_{m+1} - 2H_{m+1} = W + l_{m+1} - H_{m+1} \geq 2H + 1 - H_{m+1} \geq 0.$$

The inequality needed for continuation is therefore preserved at every level. After level k the width is $W + S_k - S_m = S_k - \delta$. Every represented sum lies in $[0, S_m]$ initially, so $\delta \geq 0$; and

$$S_{n-1} = (\alpha + \beta)(2^n - 1) + O(n)$$

gives (2.3).

For the initial band, take $X = \text{FS}(\{a_j, b_j : j \leq m\})$. It has width S_m and some finite gap bound. Since

$$\frac{S_m}{H_m} \rightarrow \frac{2(\alpha + \beta)}{\max(\alpha, \beta)} > 2,$$

it satisfies the required width inequality for all sufficiently large m . $\square$

Lemma 2.4 (Gap reduction). *Let $g \geq 2$ and let X be a represented g -band. Suppose Z_0, Z_1 are disjoint sets of available terms, disjoint also from the terms used to represent X , and*

$$\sum Z_0 = z, \quad \sum Z_1 = z + 1.$$

Then

$$Y = (X + z) \cup (X + z + 1)$$

is a represented $(g - 1)$ -band of width $\text{width}(X) + 1$. Each representation consists of a subset representing an element of X , together with Z_0 or Z_1 .

Proof. Translate by $-z$. For consecutive elements $x < x'$ of X , the new set $X \cup (X + 1)$ contains $x + 1$. A gap of length $d \geq 2$ is therefore divided into gaps of lengths at most 1 and $d - 1$; gaps of length one remain harmless. The new endpoints are $\min X$ and $\max X + 1$, so all gaps are at most $g - 1$ and the width increases by one. The disjointness assumptions give the asserted representations. $\square$

2.2. Normalisation. Suppose $\alpha/\beta \notin 2^{\mathbb{Z}}$. Deleting an initial segment of a sequence multiplies its coefficient by a power of two. Such deletions, together with interchanging the sequences, arrange that

$$1 < \rho := \beta/\alpha < 2. \quad (2.4)$$

The quantities $2^j(\beta - \alpha)$ and $2^j(2\alpha - \beta)$ then tend to infinity. After a further common deletion, either choice of rounding gives

$$a_j < b_j < a_{j+1} \quad (j \geq 0). \quad (2.5)$$

Thus all terms are distinct as integers. The deletion can also place every remaining term above any prescribed finite set. To prove strong completeness of $E_{\alpha, \beta}^-$ or $E_{\alpha, \beta}^+$, it consequently suffices to prove completeness after these deletions.

2.3. The necessary conditions.

Proposition 2.5. *Let $\alpha, \beta > 0$.*

- (a) *If both coefficients are dyadic rationals, none of $E_{\alpha, \beta}^-$, $E_{\alpha, \beta}^+$ and $E_{\alpha, \beta}^\pm$ is strongly complete.*
- (b) *If $\alpha/\beta \in 2^{\mathbb{Z}}$, neither $E_{\alpha, \beta}^-$ nor $E_{\alpha, \beta}^+$ is strongly complete.*

Proof. (a) If $\alpha = m/2^r$, then $2^n\alpha$ is an even integer for $n > r$; the same holds for β . Every set in (a) therefore has only finitely many odd elements. Deleting them leaves only even subset sums.

(b) Interchanging the coefficients if necessary, write $\beta = 2^k\alpha$ with $k \geq 0$. The second floor sequence is a tail of the first, and the same is true for ceilings. After a finite deletion the ordinary value set is therefore $\{u_j : j \geq 0\}$, where

$$u_j = \lfloor \xi 2^j \rfloor \quad \text{or} \quad u_j = \lceil \xi 2^j \rceil \quad \text{for some } \xi \geq 2.$$

We show that neither tail is complete.

For floors, write $u_{j+1} = 2u_j + \varepsilon_j$ with $\varepsilon_j \in \{0, 1\}$. Induction gives

$$u_n - \sum_{j < n} u_j = u_0 + \sum_{j < n} \varepsilon_j \geq 2.$$

Hence $1 + \sum_{j < n} u_j$ is larger than every sum using only earlier terms and smaller than every term of index at least n . It is not represented, and these omitted integers tend to infinity.

For ceilings, $u_n \geq \xi 2^n \geq 2^{n+1}$. A subset sum below u_n uses only the n preceding terms, so at most 2^n nonnegative integers below u_n are represented. At least $u_n - 2^n \geq 2^n$ are omitted. Their number is unbounded, which excludes completeness. $\square$

3. SIGNED BINARY EXPANSIONS

For $y > 0$ and $i \geq 0$, put

$$u_i^-(y) = \lfloor 2^i y \rfloor, \quad u_i^+(y) = \lceil 2^i y \rceil, \quad F_y^\pm(K) = \sum_{i \geq 0} \text{bit}_i(K) u_i^\pm(y) \quad (K \geq 1).$$

Thus $F_y^\pm(K)$ is the rounded sum selected by the ordinary binary digits of K . Define the floor and ceiling carries by

$$\varepsilon_i^-(y) = u_{i+1}^-(y) - 2u_i^-(y), \quad \varepsilon_i^+(y) = 2u_i^+(y) - u_{i+1}^+(y). \quad (3.1)$$

Both belong to $\{0, 1\}$. For either rounding convention, eventual vanishing of the carries is equivalent to y being dyadic rational. Indeed, vanishing from index i_0 gives $u_i^\pm(y) = 2^{i-i_0} u_{i_0}^\pm(y)$ for every $i \geq i_0$; division by 2^i and passage to the limit yield $y = 2^{-i_0} u_{i_0}^\pm(y)$. The converse follows directly when $y = m/2^r$. In particular, a nondyadic coefficient has infinitely many nonzero carries for each convention.

Lemma 3.1 (Signed carry intervals). *Let $K, H \geq 1$ be integers with $2^H > K$, and put $v = v_2(K)$. Among the signed expansions*

$$K = \sum_{i=v}^H \sigma_i 2^i, \quad \sigma_i \in \{-1, 0, 1\},$$

the rounded sums $\sum_{i=v}^H \sigma_i u_i^\pm(y)$ realize every integer of the respective intervals

$$I_{K,H}^-(y) = [F_y^-(K), u_H^-(y) - F_y^-(2^H - K)], \quad (3.2)$$

$$I_{K,H}^+(y) = [u_H^+(y) - F_y^+(2^H - K), F_y^+(K)]. \quad (3.3)$$

They contain $\lfloor Ky \rfloor$ and $\lceil Ky \rceil$, respectively, and have widths

$$\text{width } I_{K,H}^\pm(y) = \sum_{i=v}^{H-1} \varepsilon_i^\pm(y). \quad (3.4)$$

The ordinary binary expansion realizes the lower floor endpoint and the upper ceiling endpoint, using only the nonzero digit positions of K .

Proof. First suppose K is odd. Write $K = \sum_{i=0}^H k_i 2^i$, with $k_0 = 1$ and $k_H = 0$. For bits $c_1, \dots, c_H$, set $c_0 = c_{H+1} = 0$ and

$$\sigma_i = k_i + c_i - 2c_{i+1} \quad (0 \leq i \leq H).$$

The weighted sum telescopes to K . For $1 \leq i < H$, the digit σ_i belongs to $\{-1, 0, 1\}$ exactly when

$$c_{i+1} \leq c_i \quad \text{if } k_i = 0, \quad c_i \leq c_{i+1} \quad \text{if } k_i = 1.$$

There is no additional restriction at $i = 0$ or $i = H$. These relations orient the path on $c_1, \dots, c_H$. Starting with all coordinates zero, set them to one in an order that respects the indicated predecessors. Such an order exists because an oriented path has no directed cycle. Every intermediate vector is admissible.

The rounded sums along this path are

$$\begin{aligned}\sum_i \sigma_i u_i^-(y) &= F_y^-(K) + \sum_{j=1}^H c_j \varepsilon_{j-1}^-(y), \\ \sum_i \sigma_i u_i^+(y) &= F_y^+(K) - \sum_{j=1}^H c_j \varepsilon_{j-1}^+(y).\end{aligned}$$

Each step therefore changes the value by zero or one, monotonically in the appropriate direction. Every integer between the endpoints occurs.

At the all-ones endpoint the signed digits are $\sigma_0 = -1$, $\sigma_i = k_i - 1$ for $1 \leq i < H$, and $\sigma_H = 1$. The resulting exact identity is

$$K = 2^H - 1 - \sum_{i=1}^{H-1} (1 - k_i) 2^i.$$

The expression $1 + \sum_{i=1}^{H-1} (1 - k_i) 2^i$ is the ordinary binary expansion of $2^H - K$. Hence the rounded endpoint is $u_H^\pm(y) - F_y^\pm(2^H - K)$, and its distance from the ordinary endpoint is the sum of the carries. This proves the interval and width formulas for odd K . For $K = 2^v k$ with k odd, apply the same argument to k , $2^v y$, and $H - v$; insert zero digits below v .

It remains to locate the rounded multiple. A sum of floors is at most the floor of the sum, so $F_y^-(K) \leq \lfloor Ky \rfloor$, while

$$u_H^-(y) - F_y^-(2^H - K) \geq \lfloor 2^H y \rfloor - \lfloor (2^H - K)y \rfloor \geq \lfloor Ky \rfloor.$$

For ceilings, $F_y^+(K) \geq \lceil Ky \rceil$ and

$$u_H^+(y) - F_y^+(2^H - K) \leq \lceil 2^H y \rceil - \lceil (2^H - K)y \rceil \leq \lceil Ky \rceil.$$

These establish the asserted containment. $\square$

For example, take $K = 3$, $H = 3$, and $y = \sqrt{2}$. The admissible carry vectors

$$(0, 0, 0), \quad (0, 1, 0), \quad (1, 1, 0), \quad (1, 1, 1)$$

give the signed expansions $3 = 1 + 2 = 1 - 2 + 4 = -1 + 4 = -1 - 4 + 8$. Since $(u_0^-, u_1^-, u_2^-, u_3^-) = (1, 2, 5, 11)$, their rounded values are 3, 4, 4, 5. Thus every integer of $I_{3,3}^-(\sqrt{2}) = [3, 5]$ occurs; the zero increment reflects a vanishing carry.

Write

$$h(K) = \lfloor \log_2 K \rfloor + 1, \quad I_K^\pm(y) = I_{K, h(K)}^\pm(y).$$

The top index $h(K)$ is the least allowed in the lemma. An integer interval is *nondegenerate* if it contains at least two integers.

Lemma 3.2 (A unit). *Let P, Q be positive integers and let $e = |Q\alpha - P\beta| > 0$. Suppose $n_0 \geq 0$ satisfies $1/2 \leq 2^{n_0} e < 1$, and suppose the two rays have distinct integer values and use the same rounding convention. For some $n \in \{n_0, n_0 + 1\}$ there are disjoint blocks Z_0, Z_1 of terms, supported in $[n, n + h]$ with $h = h(\max(P, Q))$, such that $\sum Z_1 = \sum Z_0 + 1$. Moreover*

$$\frac{1}{2e} \leq 2^n < \frac{2}{e}. \quad (3.5)$$

Proof. Put $y = 2^{n_0}\alpha$ and $z = 2^{n_0}\beta$. For floors, the intervals $I_Q^-(y)$ and $I_P^-(z)$ contain $\lfloor Qy \rfloor$ and $\lfloor Pz \rfloor$. These integers are equal or adjacent because $|Qy - Pz| < 1$. If adjacent, use their signed realizations. If equal and either interval is nondegenerate, the intervals again contain adjacent integers, one in each. The only exception is a common singleton $\{\kappa\}$. Then write $Qy = \kappa + u$ and $Pz = \kappa + v$, with $u, v \in [0, 1)$. Since $|u - v| \geq 1/2$, $\lfloor 2u \rfloor$ and $\lfloor 2v \rfloor$ differ by one. Therefore the anchors of $I_Q^-(2y)$ and $I_P^-(2z)$ at the next level are adjacent.

For ceilings the same argument starts from $\lceil Qy \rceil$ and $\lceil Pz \rceil$. In the common-singleton case, write $Qy = \kappa - u$ and $Pz = \kappa - v$, with $u, v \in [0, 1)$. The next anchors are $2\kappa - \lfloor 2u \rfloor$ and $2\kappa - \lfloor 2v \rfloor$, which again differ by one.

In either convention, select the resulting adjacent signed sums and subtract the smaller from the larger. Move every negative term to the other side. Each term has coefficient in $\{-1, 0, 1\}$ and occurs in at most one ray, so the positive and negative supports are disjoint subsets of the available terms. They are the required blocks Z_1, Z_0 . Their indices lie in $[n, n + h]$, and the choice $n \in \{n_0, n_0 + 1\}$ gives (3.5). $\square$

4. IRRATIONAL RATIOS

Assume throughout that the rays have the same rounding convention and have been normalised so that $1 < \rho = \beta/\alpha < 2$ and $a_j < b_j < a_{j+1}$, with ρ irrational. The unit lemma reduces the gap bound; later terms restore the continuation inequality. The unit and the restoring translates must be chosen at compatible scales.

4.1. Two scales. After omitting any initial repeated denominator, let Q_k/P_k be the continued-fraction convergents of ρ , with strictly increasing positive denominators. They satisfy

$$\frac{1}{P_{k+1} + P_k} < |P_k\rho - Q_k| < \frac{1}{P_{k+1}}, \quad (4.1)$$

and $\gcd(P_k, Q_k) = 1$: if $\tau = a_{k+1} + 1/\tau'$ with $\tau' > 1$ is the complete quotient following a_{k+1} , then $\rho = (Q_k\tau + Q_{k-1})/(P_k\tau + P_{k-1})$ and $|P_k\rho - Q_k| = 1/(P_k\tau + P_{k-1}) = 1/(P_{k+1} + P_k/\tau')$. In particular $|\rho - Q_k/P_k| < 1/P_k^2$.

Let Q/P be a sufficiently large convergent with successor denominator P^+ , and apply Lemma 3.2 with $e = |Q\alpha - P\beta| = \alpha|P\rho - Q|$. By (4.1) and (3.5), the level n of the unit satisfies

$$\frac{P^+}{2\alpha} < 2^n < \frac{4P^+}{\alpha}. \quad (4.2)$$

Lemma 4.1 (Joint choice of scales). *There are convergents Q/P of ρ with P arbitrarily large, later convergent denominators q , and levels n of the unit of Lemma 3.2 for Q/P , such that*

$$q \geq 128P, \quad \frac{\alpha}{4}2^n < q < 256\alpha 2^n. \quad (4.3)$$

In particular $\log q = O(n)$ along these choices.

Proof. Let q be a large convergent denominator and let P be the last convergent denominator with $P \leq q/128$; its successor P^+ satisfies $q/128 < P^+ \leq q$. By (4.2),

$$\frac{q}{256\alpha} < \frac{P^+}{2\alpha} < 2^n < \frac{4P^+}{\alpha} \leq \frac{4q}{\alpha}.$$

Both P and n tend to infinity with q ; no bound on P^+/P is needed. $\square$

Lemma 4.2 (A rectangle of nominal sums). *Let $A > 0$, $B = \rho A$, and let q be a convergent denominator of ρ . For an integer $K \geq 16q$ put $m_0 = \lceil \rho(q - 1) \rceil$. The set $\{Au + Bv : u, v \in \mathbb{Z}, 0 \leq u, v \leq K\}$ contains a subset with least element Am_0 , greatest element $AK + B(K - q + 1)$, consecutive gaps at most $3A/q$, and hence span at least*

$$A((1 + \rho)K - 4q - 1). \quad (4.4)$$

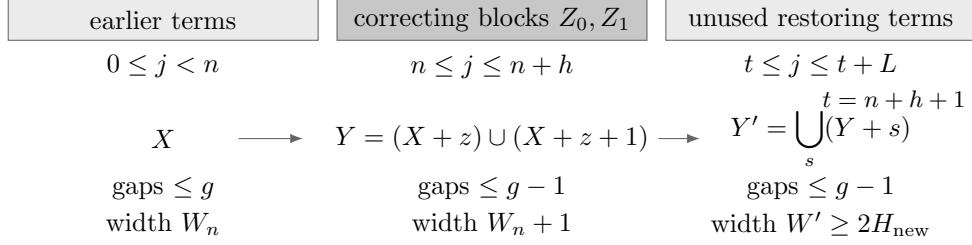


FIGURE 1. One round of gap reduction and restoration for an irrational ratio. The sums of Z_0 and Z_1 are z and $z + 1$, each s is a subset sum of the restoring terms, and H_{new} is the largest term at the final level. In every representation the three support blocks are disjoint; overlapping translates preserve the reduced gap bound, and the restored width permits continuation.

Proof. Let p/q be the convergent. The points vp/q , $0 \leq v < q$, are the full grid of mesh $1/q$ on $\mathbb{R}/\mathbb{Z}$, and $|v\rho - vp/q| < v/q^2 < 1/q$, so the points $\{v\rho\}$, $0 \leq v < q$, leave no arc of $\mathbb{R}/\mathbb{Z}$ of length greater than $3/q$ empty.

Work in units of A . For $m_0 \leq m \leq K - 1$ and $0 \leq v < q$ put $u = m - \lfloor \rho v \rfloor$; then $0 \leq u \leq K$ and $u + \rho v = m + \{\rho v\}$. Together with the point K (from $u = K$, $v = 0$) these points cover $[m_0, K]$ with consecutive gaps at most $3/q$. Now translate this set by ρv_0 for $0 \leq v_0 \leq K - q + 1$: the second coordinate becomes $v + v_0 \leq K$, so the translates stay in the rectangle, and the spanning intervals of successive translates overlap because $K - m_0 > \rho$. The union has least element m_0 , greatest element $K + \rho(K - q + 1)$, gaps at most $3/q$, and span at least $(1 + \rho)K - 2\rho(q - 1) - 1 \geq (1 + \rho)K - 4q - 1$. $\square$

Lemma 4.3 (Stability under rounding). *Let $x_1 < \dots < x_r$ be real numbers with consecutive differences at most d , and let $y_1, \dots, y_r$ satisfy $|y_i - x_i| \leq D$. Then the distinct values among the y_i , in increasing order, have consecutive differences at most $d + 2D$ and span at least $x_r - x_1 - 2D$.*

Proof. Suppose $a < b$ are consecutive distinct values among the y_i with $b - a > d + 2D$. Some x_j is at most $a + D$ (the one corresponding to a) and some x_k is at least $b - D$ (the one corresponding to b); let x_l be the first x exceeding $a + D$. Then $x_l \leq a + D + d < b - D$, so $y_l \in (a, b)$, a contradiction. The endpoints give the span. $\square$

4.2. One round.

Theorem 4.4 (Restatement of Theorem D). *If $\alpha, \beta > 0$ and α/β is irrational, then $E_{\alpha, \beta}^-$ and $E_{\alpha, \beta}^+$ are strongly complete.*

Proof. Fix an arbitrary finite set of values to be deleted. The normalisation in Section 2.2 gives positive, pairwise distinct remaining terms, all larger than the deleted values. By Lemma 2.3, an initial prefix supports a represented g -band X of width at least $2H_m$. If $g = 1$, continuation represents every integer from $\min X$ onward. It therefore suffices to replace $g \geq 2$ by $g - 1$ while restoring the same width condition.

Gap reduction. Choose q, P, Q, n by Lemma 4.1, with q large enough that $n > m$ and all the eventual inequalities below hold. Continue X through level $n - 1$, retaining the name X . Its width is $W_n = (\alpha + \beta)2^n + O(n)$. By Lemma 3.2, disjoint blocks Z_0, Z_1 supported in $[n, n + h]$, where $h = h(\max(P, Q))$, have sums $z, z + 1$. Thus

$$Y = (X + z) \cup (X + z + 1)$$

is a represented $(g - 1)$ -band of width $W_n + 1$ by Lemma 2.4. For the restoring terms put

$$t = n + h + 1, \quad A = \alpha 2^t, \quad B = \beta 2^t = \rho A.$$

All terms of index at least t are unused. Let L be least with $K := 2^{L+1} - 1 \geq 16q$; then $K < 32q + 1$ and $L = O(\log q) = O(n)$.

Overlapping translates. For $0 \leq u, v \leq K$, their ordinary binary expansions define the actual subset sum

$$s(u, v) = \sum_{i=0}^L \text{bit}_i(u) a_{t+i} + \sum_{i=0}^L \text{bit}_i(v) b_{t+i}.$$

Each of its at most $2(L+1)$ terms differs by less than one from the corresponding real term. With $D = 2(L+1)$,

$$|s(u, v) - (Au + Bv)| \leq D. \quad (4.5)$$

Apply [Lemma 4.2](#) to A , B , q and K , and realise each point $Au + Bv$ of the subset it provides by $s(u, v)$. For large P the convergent satisfies $1 < Q/P < 2$, so $2^{h+1} \leq 4 \max(P, Q) < 8P$, and with $q \geq 128P$,

$$\frac{3A}{q} = \frac{3\alpha 2^{h+1}}{q} 2^n < \frac{3\alpha}{16} 2^n.$$

By [Lemma 4.3](#) the distinct values of $s(u, v)$ so obtained have consecutive gaps at most

$$\frac{3A}{q} + 2D < W_n + 1 \quad (4.6)$$

for q large, since $W_n = (\alpha + \beta)2^n + O(n)$ and $D = O(n)$. The translates $Y + s(u, v)$ therefore satisfy the hypothesis of [Lemma 2.2](#), and their union Y' is a represented $(g-1)$ -band.

Restoring the width. The largest available term through the new level is b_{t+L} , and $2b_{t+L} \leq B(K+1) + 2$, where the constant 2 is unnecessary for floors. By (4.4), the nominal span exceeds $B(K+1)$ by at least

$$A((1+\rho)K - 4q - 1) - \rho A(K+1) = A(K - 4q - 1 - \rho) \geq A(12q - 3),$$

which exceeds $2D + 2$ for q large. By [Lemma 4.3](#) the span of the values $s(u, v)$, and hence the width of Y' , exceeds $2b_{t+L}$. So Y' is a represented $(g-1)$ -band on the terms with index at most $t+L$ satisfying the hypothesis of [Lemma 2.3](#).

Disjointness and termination. Each representation in Y' uses three disjoint index ranges:

$$\underbrace{[0, n-1]}_{\text{continued band}}, \quad \underbrace{[n, n+h]}_{\text{one unit block}}, \quad \underbrace{[n+h+1, n+h+1+L]}_{\text{one restoring subset}}.$$

Each term occurs at most once, and numerical interleaving makes the summands distinct integers. Alternative representations may share terms. We have therefore constructed a represented $(g-1)$ -band satisfying continuation after the prescribed deletion. Repeat until $g = 1$, then continue indefinitely. This proves completeness after the prescribed deletion, for both rounding conventions. $\square$

5. ROOTS OF TWO

Throughout this section $k \geq 2$, $t > 0$, $\theta = \theta_k = 2^{1/k}$, and for $0 \leq r < k$ and $j \geq 0$

$$a_{r,j}^- = \lfloor t\theta^{r+kj} \rfloor = \lfloor t2^{r/k} 2^j \rfloor, \quad a_{r,j}^+ = \lceil t\theta^{r+kj} \rceil, \quad (5.1)$$

so that the terms of $E_{k,t}^\pm(R)$ are the $a_{r,j}^\pm$ with $r \in R$. For fixed r the sequence $j \mapsto a_{r,j}^\pm$ is the dyadic sequence with coefficient $t2^{r/k}$.

Lemma 5.1. *For $1 \leq d < k$ the number $2^{d/k}$ is irrational. Consequently, for $0 \leq r < s < k$ the ratio $t2^{s/k}/t2^{r/k}$ is irrational, and at most one of the k coefficients $t2^{r/k}$, $0 \leq r < k$, is a dyadic rational.*

Proof. If $2^{d/k} = a/b$ with coprime positive integers a, b , then $a^k = 2^d b^k$; coprimality forces $b = 1$, unique factorisation forces $a = 2^m$ for an integer $m \geq 0$, and then $km = d$, which is impossible for $0 < d < k$. The ratio of two coefficients is $2^{(s-r)/k}$ with $0 < s - r < k$; if two coefficients were dyadic rationals their ratio would be rational. $\square$

Lemma 5.2. *There is j_1 such that for all $j \geq j_1$, and for either rounding,*

$$0 < a_{0,j} < a_{1,j} < \cdots < a_{k-1,j} < a_{0,j+1}. \quad (5.2)$$

Moreover, for every $j \geq 0$ and $r < k$ the values $a_{r',j'}$ with $r' + kj' < r + kj$ are at most $a_{r,j}$, and those with $r' + kj' > r + kj$ are at least $a_{r,j}$.

Proof. The real numbers $t\theta^{r+kj}$ increase strictly with $r + kj$, and rounding is monotone, which gives the last assertion. Consecutive real numbers $t\theta^{r+kj}$ and $t\theta^{r+kj+1}$ differ by $t\theta^{r+kj}(\theta - 1)$, which tends to infinity with j ; once it exceeds 2 the rounded values, each within 1 of the real number it rounds, are in the same strict order, and the first is positive once $t\theta^{kj} \geq 1$. $\square$

Proof of Theorem A and of the sufficiency in Theorem B(a). Let $r < s$ be in R , and put $\alpha = t2^{r/k}$, $\beta = t2^{s/k}$. By (5.1), the terms of $E_{k,t}^-(\{r, s\})$ are the values $\lfloor 2^j \alpha \rfloor, \lfloor 2^j \beta \rfloor$, $j \geq 0$, other than 0, and those of $E_{k,t}^+(\{r, s\})$ the values $\lceil 2^j \alpha \rceil, \lceil 2^j \beta \rceil$; and $\beta/\alpha = 2^{(s-r)/k}$ is irrational by Lemma 5.1. By Theorem D both sets are strongly complete. If a set E contains a strongly complete set E_0 , then for every finite $F \subseteq E$ the set $E \setminus F$ contains $E_0 \setminus (F \cap E_0)$, which is complete; so E is strongly complete. This proves the sufficiency in Theorem B(a) and, with $R = \{0, \dots, k-1\}$, the first assertion of Theorem A. For the second assertion delete from $E_{k,t}^-$ every value at most

$$\max(\{B_0\} \cup \{\lfloor t2^{n/k} \rfloor : 0 \leq n < n_0\});$$

every remaining value exceeds B_0 , and, by the monotonicity in Lemma 5.2, is attained only at indices $n \geq n_0$. The ceiling case is the same with ceilings in the finite maximum. $\square$

5.1. One dyadic sequence.

Proposition 5.3. *For $c > 0$ put $\mathcal{D}^-(c) = \{\lfloor c2^j \rfloor : j \geq 0\} \setminus \{0\}$ and $\mathcal{D}^+(c) = \{\lceil c2^j \rceil : j \geq 0\}$.*

- (a) *Neither $\mathcal{D}^-(c)$ nor $\mathcal{D}^+(c)$ is strongly complete.*
- (b) *$\mathcal{D}^-(c)$ is complete if and only if $c = 2^{-m}$ for an integer $m \geq 0$.*

Proof. (a) Deleting finitely many values leaves $\{u_j : j \geq 0\}$ with $u_j = \lfloor \xi 2^j \rfloor$ or $u_j = \lceil \xi 2^j \rceil$ for some $\xi \geq 2$, and it suffices to show that this set is not complete. The u_j are strictly increasing, since $u_{j+1} \geq 2u_j - 1 > u_j$ for $u_j \geq 2$. In the floor case write $u_{j+1} = 2u_j + \varepsilon_j$ with $\varepsilon_j \in \{0, 1\}$. Then, by induction on n ,

$$u_n - \sum_{j < n} u_j = u_0 + \sum_{j < n} \varepsilon_j, \quad (5.3)$$

which is at least $u_0 \geq 2$, so the integer $1 + \sum_{j < n} u_j$ exceeds every sum of terms of index below n and is smaller than u_n ; it is not a subset sum, and this holds for every n . In the ceiling case $u_n \geq \xi 2^n \geq 2^{n+1}$, while a subset sum below u_n uses only the n terms of index below n , so at most 2^n integers below u_n are subset sums; at least $u_n - 2^n \geq 2^n$ integers below u_n are missing, which is unbounded.

(b) Let m be the first index with $\lfloor c2^m \rfloor \geq 1$ and put $v_j = \lfloor c2^{m+j} \rfloor$; these are all the positive values, without repetition, and $v_{j+1} = 2v_j + \varepsilon_j$ with $\varepsilon_j \in \{0, 1\}$, so that $v_n - \sum_{j < n} v_j = v_0 + \sum_{j < n} \varepsilon_j$ as in (5.3). If $v_0 \geq 2$, or if some $\varepsilon_j = 1$, then $1 + \sum_{j < n} v_j$ is missing for every sufficiently large n , and these missing integers tend to infinity. Otherwise $v_j = 2^j$ for every j , so $1 \leq c2^m < 1 + 2^{-j}$ for all j , forcing $c = 2^{-m}$; conversely such a coefficient gives exactly the powers of two, which represent every nonnegative integer. $\square$

Proof of Theorem B, completion. For $R = \emptyset$ the sets are empty. For $R = \{r\}$ they are $\mathcal{D}^-(t2^{r/k})$ and $\mathcal{D}^+(t2^{r/k})$, and Proposition 5.3(a) gives the necessity in (a), while Proposition 5.3(b) gives the single-class case of (b). For $|R| \geq 2$ strong completeness, already proved, implies completeness. $\square$

Remark 5.4. Theorem B(b) concerns floors. For a ceiling sequence $u_j = \lceil c2^j \rceil$, the deficit $u_n - \sum_{j < n} u_j = u_0 - \sum_{j < n} \varepsilon_j$ is nonincreasing, and a wider range of coefficients gives completeness. If $0 < c \leq 1$, discard all but the last initial occurrence of 1. The resulting distinct sequence starts with 1 and has $u_{j+1} \leq 2u_j$. Induction gives $u_{j+1} \leq 2u_j \leq 1 + \sum_{i \leq j} u_i$, so its prefix subset sums fill intervals from 0. In contrast, for $c = 3$ every term is a multiple of 3, and for $c = 3/2$ the values 2, 3, 6, 12, ... miss every integer congruent to 1 modulo 3.

5.2. Bases outside the golden interval. Write $\phi = (1 + \sqrt{5})/2$.

Proposition 5.5. *Let $t > 0$ and $\alpha > 0$ with $\alpha \notin (1, 2)$. Neither the set of positive values $\lfloor t\alpha^n \rfloor$ nor the set of values $\lceil t\alpha^n \rceil$ is strongly complete. If $\alpha > 2$, the number of subset sums of either set that are at most X is $O_{t,\alpha}(X^{\log_\alpha 2})$, and neither set is complete.*

Proof. For $0 < \alpha \leq 1$ the set of values is finite. For $\alpha = 2$ the sets are $\mathcal{D}^-(t)$ and $\mathcal{D}^+(t)$, and Proposition 5.3(a) applies. For $\alpha > 2$ a term at most X has $t\alpha^n \leq X + 1$, so at most $\log_\alpha X + O_{t,\alpha}(1)$ terms are at most X ; a subset sum at most X uses only such terms, so there are at most $2^{\log_\alpha X + O(1)} = O(X^{\log_\alpha 2})$ of them, and $\log_\alpha 2 < 1$. $\square$

Lemma 5.6 (Propagation of a gap). *Let $b_1 < b_2 < \dots$ be positive integers with $b_j + b_{j+1} \leq b_{j+2}$ for all $j \geq 1$, and put $S_r = \sum_{j \leq r} b_j$, $S_0 = 0$. If $S_r < m < b_{r+1}$ for integers $r \geq 0$ and m , then the integers $m_k = m + \sum_{i=1}^{k-1} b_{r+2i+1}$, $k \geq 1$, are not sums of distinct terms b_j ; in particular the set $\{b_j : j \geq 1\}$ is not complete.*

Proof. We show by induction on k that m_k is missing and $S_{r+2k-2} < m_k < b_{r+2k}$. For $k = 1$ the terms of index at most r have total less than m , while every later term exceeds m ; so $m_1 = m$ is missing, and both bounds hold. Suppose the claim holds for k , and put $v = b_{r+2k+1}$. The growth assumption gives $S_{r+2k} = S_{r+2k-2} + b_{r+2k-1} + b_{r+2k} \leq S_{r+2k-2} + v < m_k + v$, and $m_k + v < b_{r+2k} + b_{r+2k+1} \leq b_{r+2k+2}$. A representation of $m_k + v$ must use v , since without it the usable terms total at most S_{r+2k} and every later term exceeds the target; removing v would represent m_k . Hence $m_{k+1} = m_k + v$ is missing, and the bounds hold. The m_k tend to infinity. $\square$

Proposition 5.7. *If $E(t, \alpha)$ is strongly complete for some $t > 0$, then $1 < \alpha < \phi$.*

Proof. By Proposition 5.5 it remains to exclude $\phi \leq \alpha < 2$. Put $u_n = \lfloor t\alpha^n \rfloor$. Since $1 + \alpha \leq \alpha^2$,

$$u_n + u_{n+1} \leq \lfloor t\alpha^n(1 + \alpha) \rfloor \leq u_{n+2} \quad (n \geq 0),$$

and once a value is at least 2 its real consecutive gap $t\alpha^n(\alpha - 1)$ exceeds $2(\alpha - 1) > 1$, so all later values are distinct. Delete every value before a raw index n_1 so late that $u_{n_1} \geq 2$; the retained values $b_1 < b_2 < \dots$ satisfy $b_j + b_{j+1} \leq b_{j+2}$, and Lemma 5.6 with $r = 0$, $m = 1$ shows that they are not complete. This finite deletion excludes strong completeness. $\square$

6. DISJOINT SUBSETS

An unbounded set of nonnegative integers has *bounded gaps* if the differences between consecutive elements are bounded. The set of subset sums of a complete set has bounded gaps; the converse fails, for instance for the set of even numbers.

Lemma 6.1 (Carry budget). *Let E be an infinite set of positive integers such that $\text{FS}(E)$ has gaps at most g . For a real $x > 0$ let T be the sum of the elements of E that are less than x , and let y be the least element of E that is at least x . Then $y - T \leq g$.*

Proof. If $y \leq T$ there is nothing to prove. Otherwise T and y are elements of $\text{FS}(E)$ and no element of $\text{FS}(E)$ lies strictly between them: a sum that uses an element of E at least x is at least y , and a sum that uses none is at most T . Hence $y - T \leq g$. $\square$

Proof of Theorem C. No k disjoint subsets. Write $a_{r,j} = a_{r,j}^-$ and $\varepsilon_{r,j} = a_{r,j+1} - 2a_{r,j} \in \{0, 1\}$ for $0 \leq r < k$. Summing $a_{r,j+1} = 2a_{r,j} + \varepsilon_{r,j}$ over $j < n$ gives $a_{r,n} = 2^n a_{r,0} + \sum_{j < n} 2^{n-1-j} \varepsilon_{r,j}$, and summing this over j and then over r gives the identity

$$\sum_{r < k} a_{r,n} - \sum_{j < n} \sum_{r < k} a_{r,j} = \sum_{r < k} a_{r,0} + \sum_{j < n} \sum_{r < k} \varepsilon_{r,j}. \quad (6.1)$$

By Lemma 5.1 at least $k-1 \geq 1$ of the coefficients $t2^{r/k}$ are not dyadic rationals, and each of these has infinitely many nonzero carries (Section 3); so the right-hand side of (6.1) tends to infinity with n .

Suppose that $E_1, \dots, E_k \subseteq E_{k,t}^-$ are pairwise disjoint and that each $\text{FS}(E_i)$ is unbounded with gaps at most g_i ; each E_i is then infinite. Let j_1 be as in Lemma 5.2, let $n > j_1$, and take $x = a_{0,n}$ in Lemma 6.1: let T_i be the sum of the elements of E_i below $a_{0,n}$ and x_i the least element of E_i at least $a_{0,n}$. By Lemma 5.2 the elements of $E_{k,t}^-$ that are at least $a_{0,n}$ are the values $a_{r,j}$ with $r + kj \geq kn$, and the k smallest of them are $a_{0,n} < a_{1,n} < \dots < a_{k-1,n}$; since the x_i are distinct,

$$\sum_i x_i \geq \sum_{r < k} a_{r,n}.$$

The sets E_i are disjoint, so $\sum_i T_i$ is at most the sum of the distinct elements of $E_{k,t}^-$ below $a_{0,n}$, which are among the values $a_{r,j}$ with $r + kj < kn$; hence

$$\sum_i T_i \leq \sum_{j < n} \sum_{r < k} a_{r,j},$$

a sum in which a value attained at several indices is counted several times. By Lemma 6.1 and (6.1),

$$\sum_i g_i \geq \sum_i (x_i - T_i) \geq \sum_{r < k} a_{r,0} + \sum_{j < n} \sum_{r < k} \varepsilon_{r,j},$$

and the right-hand side is unbounded in n , a contradiction.

$\lfloor k/2 \rfloor$ *disjoint strongly complete subsets.* The sets $E_{k,t}^\pm(\{2i, 2i+1\})$, $0 \leq i < \lfloor k/2 \rfloor$, are strongly complete by Theorem B(a). Delete from each the values $a_{r,j}^\pm$ with $j < j_1$; by (5.2) the remaining values of distinct classes are distinct, so the $\lfloor k/2 \rfloor$ sets become pairwise disjoint, and each remains strongly complete.

k *disjoint ceiling subsets with bounded gaps.* Delete from each ceiling class the values with $j < j_1$; the k classes are then pairwise disjoint. A single class is $\{u_j : j \geq j_1\}$ with $u_j = \lceil \xi 2^j \rceil$, $\xi = t2^{r/k}$, and $u_{j+1} = 2u_j - \varepsilon_j$ with $\varepsilon_j \in \{0, 1\}$, so by induction

$$u_n - \sum_{j_1 \leq j < n} u_j = u_{j_1} - \sum_{j_1 \leq j < n} \varepsilon_j \leq u_{j_1}.$$

We show by induction on n that $\text{FS}(\{u_j : j_1 \leq j \leq n\})$ has consecutive gaps at most u_{j_1} . For $n = j_1$ the set is $\{0, u_{j_1}\}$. Passing from $n-1$ to n replaces the set Y by $Y \cup (Y + u_n)$, and $\min(Y + u_n) = u_n \leq \sum_{j_1 \leq j < n} u_j + u_{j_1} = \max Y + u_{j_1}$, so Lemma 2.2 applies with $g = u_{j_1}$. The union over n inherits the bound. *Strong completeness forces* $1 < \alpha < \phi$. This is Proposition 5.7. $\square$

Corollary 6.2. *For every $t > 0$ the set $E_{2,t}^-$ is strongly complete but contains no two disjoint subsets whose subset sums are unbounded with bounded gaps. For $k \geq 4$ every sufficiently large integer has $\lfloor k/2 \rfloor$ representations as a sum of distinct elements of $E_{k,t}^-$ whose sets of summands are pairwise disjoint.*

Proof. The first assertion is [Theorems A and C](#) with $k = 2$. For the second, take the largest of the completeness thresholds of the $\lfloor k/2 \rfloor$ pairwise disjoint strongly complete sets in the proof of [Theorem C](#). $\square$

7. FINITE VERIFICATION

The proof of [Theorem D](#) ends with a represented 1-band of width at least twice the largest term used, and [Lemma 2.3](#) then represents every integer from its left endpoint on, which is Brown's criterion [2] in the form of [Lemma 2.3](#). The following statement isolates what a finite computation has to check, and allows for values that are repeated among the early indices.

Proposition 7.1. *Let a_j, b_j be two dyadic sequences with the same rounding, let $J \geq 0$, and let V be a finite set of positive integers. Suppose that $0 < a_J < b_J < a_{J+1}$ and that every term of index greater than J is a positive integer outside V . If $[C, C'] \subseteq \text{FS}(V)$ with $C' - C \geq 2b_J$, then every integer $N \geq C$ is a sum of distinct elements of $V \cup \{a_j, b_j : j > J\}$.*

Proof. The interval $[C, C']$ is a represented 1-band on V of width at least $2H_J$. The interleaving propagates by the inequalities below, so the later terms are mutually distinct. The proof of [Lemma 2.3](#) uses only the recurrence (2.2) and the fact that each adjoined term lies outside the subsets used before it. Both hold here; continuation therefore represents every integer from C on. $\square$

Since the rounding errors of two floors, or of two ceilings, have the same sign and are less than 1,

$$b_{j+1} - a_{j+1} \geq 2(b_j - a_j) - 1, \quad a_{j+2} - b_{j+1} \geq 2(a_{j+1} - b_j) - 1,$$

so the interleaving $a_J < b_J < a_{J+1}$ propagates to every later level. When the two sequences are the classes $r < s$ of one sequence $\lfloor t\theta_k^n \rfloor$, or of the ceilings, every term of index greater than J exceeds every element of V by [Lemma 5.2](#); so the hypotheses of [Proposition 7.1](#) are decided by finitely many terms.

For a rational coefficient $t = c/d$ every rounded value is decided by integer arithmetic: for positive quantities

$$a = \lfloor (c/d)2^{n/k} \rfloor \iff a^k d^k \leq c^k 2^n < (a+1)^k d^k, \quad (7.1)$$

which is the definition of the integer part multiplied through by $d^k > 0$, and the ceiling is a if equality holds on the left and $a+1$ otherwise.

Proposition 7.2. *Fix $k \geq 2$, $t > 0$, two residues $0 \leq r < s < k$, a first index $n_0 \geq 0$ and a rounding, and let u_n be $\lfloor t2^{n/k} \rfloor$ or $\lceil t2^{n/k} \rceil$. For $J = 0, 1, 2, \dots$ let V_J be the set of positive values u_n with $n_0 \leq n \leq kJ + s$ and $n \bmod k \in \{r, s\}$, compute $\text{FS}(V_J)$, and, once $kJ + r \geq n_0$ and $0 < u_{kJ+r} < u_{kJ+s} < u_{k(J+1)+r}$, search $\text{FS}(V_J)$ for an interval of width at least $2u_{kJ+s}$. This search terminates, and its output satisfies the hypotheses of [Proposition 7.1](#). If t is rational, every step is a computation with integers by (7.1).*

Proof. Apply the proof of [Theorem D](#) to the two classes with all indices below n_0 deleted. It produces a represented 1-band of width at least twice the largest term at its last level, supported on the values of some V_J ; its elements belong to $\text{FS}(V_J)$, so the search succeeds at that level at the latest. $\square$

If the interval found has left endpoint C , then C is the least integer from which every integer is represented as soon as $C - 1 \notin \text{FS}(V_J)$ and $C - 1 < u_{k(J+1)+r}$, since every term outside V_J then exceeds $C - 1$. A representation of a prescribed $N \geq C$ is recovered by keeping the sets FS of the successive prefixes of the permitted terms and deciding the terms from the largest downwards: a term is omitted if the remainder is a subset sum of the preceding prefix and included otherwise, so the remainder reaches 0 with each term used at most once. The accompanying programs implement the search and this recovery.

APPENDIX A. EXACT COMPUTATIONS

Proposition A.1. *Every integer $N \geq 6$ is a sum of distinct elements of $\{\lfloor 3 \cdot 2^{n/2} \rfloor : n \geq 0\}$, and 5 is not.*

Proof. The first terms are 3, 4, 6, 8, 12, 16, 24, 33. The terms 3, 4, 6, 8 represent every integer from 6 to 15:

6	6	11	$8 + 3$
7	$3 + 4$	12	$8 + 4$
8	8	13	$6 + 4 + 3$
9	$6 + 3$	14	$8 + 6$
10	$6 + 4$	15	$8 + 4 + 3$

Adjoining 12 adds $[18, 27]$, and $16 = 12 + 4$, $17 = 8 + 6 + 3$, so the first five terms represent $[6, 27]$; adjoining 16 adds $[22, 43]$, so $[6, 43] \subseteq \text{FS}(\{3, 4, 6, 8, 12, 16\})$. The pairs through level 2 are $(3, 4), (6, 8), (12, 16)$, the width $43 - 6 = 37$ exceeds $2 \cdot 16$, and $a_3 = 24 > 16$; [Proposition 7.1](#) represents every $N \geq 6$. Below 6 only 3 and 4 are available, and they do not represent 5. $\square$

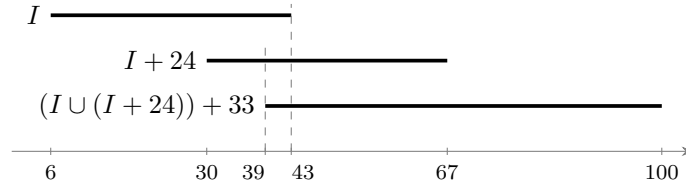


FIGURE 2. Continuation from the interval $I = [6, 43]$ of [Proposition A.1](#): adjoining 24 gives $[6, 67]$ and adjoining 33 gives $[6, 100]$, of width $94 \geq 2 \cdot 33$. All three sets are intervals of subset sums.

Proposition A.2. *Every integer $N \geq 4$ is a sum of distinct values $\lfloor 2^{n/2} \rfloor$ with $n \geq 2$, and 3 is not.*

Proof. The distinct values with $2 \leq n \leq 7$ are 2, 4, 5, 8, 11 (the value 2 occurs at $n = 2$ and $n = 3$), with sum 30. They represent every integer from 4 to 15:

4	4	10	$8 + 2$
5	5	11	11
6	$4 + 2$	12	$8 + 4$
7	$5 + 2$	13	$8 + 5$
8	8	14	$8 + 4 + 2$
9	$5 + 4$	15	$11 + 4$

Complementary subsets give $[15, 26]$, so $[4, 26]$ is represented, of width $22 = 2 \cdot 11$. The pair at level 3 is $(8, 11)$ and the next term of the first class is 16, so [Proposition 7.1](#) applies although the value 2 was retained once. The only value below 4 is 2, so 3 is missing. $\square$

[Table 1](#) lists ten thresholds found by the search of [Proposition 7.2](#); the accompanying programs reproduce them. For each row the permitted exponents are $n \geq n_0$ with $n \bmod k \in R$, J is the last level, $H_J = u_{kJ+s}$ is the largest term at that level, $[C, C'] \subseteq \text{FS}(V_J)$ with $C' - C \geq 2H_J$, and $C - 1$ is absent from $\text{FS}(V_J)$ and smaller than the next permitted term; so C is the least completeness threshold of the set in that row.

k	t	R	n_0	J	$[C, C']$	$C' - C$	$2H_J$
2	1	$\{0, 1\}$	2	3	$[4, 26]$	22	22
2	3	$\{0, 1\}$	0	2	$[6, 43]$	37	32
2	10	$\{0, 1\}$	0	5	$[160, 1359]$	1199	904
2	7/3	$\{0, 1\}$	5	9	$[986, 4743]$	3757	3378
3	3	$\{0, 2\}$	0	4	$[21, 218]$	197	152
4	3	$\{1, 3\}$	0	3	$[17, 110]$	93	80
5	10	$\{1, 4\}$	0	7	$[755, 6605]$	5850	4456
8	3	$\{2, 7\}$	0	3	$[21, 113]$	92	88
2	3	$\{0, 1\}$	0	3	$[20, 90]$	70	68
4	3	$\{1, 3\}$	0	4	$[29, 245]$	216	162

TABLE 1. Optimal completeness thresholds C for the sets $E_{k,t}^\pm(R)$ restricted to exponents $n \geq n_0$; floors in the first eight rows, ceilings in the last two.

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