

ABSTRACT. We prove that, for every sufficiently large k , the number 1 is a sum of reciprocals over k pairwise disjoint, pairwise nonadjacent intervals of positive integers, each of length two or three. This answers the question in the form in which it is posed, with the shortest admissible intervals.

We show more generally that every positive rational number is such a sum over exactly k intervals of length two or three for every sufficiently large k , with all denominators above any prescribed bound and any two intervals separated by any prescribed number of integers; for fixed rational, bound and separation there are at least $2^{k/21}$ such families with largest denominator $O(k^{2-\delta})$, and a family with $O(k^{1-\delta/4})$ triples. The construction descends through the prime-power divisors of an exact rational remainder using primitive points on modular hyperbolas, completes on a fixed reserve of pairs, and switches independently between five triples and six pairs of equal reciprocal sum; one completion serves every switch, and overlapping count intervals give every large k . Taken together, the reciprocal mass, the leading prime-power residue and the component count are controlled at once, and the separation is never traded for any of them.

1. INTRODUCTION

Erdős and Graham [8, p. 34] ask whether 1 can be written as a sum of reciprocals over exactly k intervals of positive integers, each of length at least two, for every sufficiently large k ; the question is recorded as Problem 289 at <https://www.erdosproblems.com/289>, in the form in which the intervals are distinct, pairwise disjoint and pairwise nonadjacent. Thus, after ordering them as $I_i = [a_i, b_i]$, the requirements are

$$1 = \sum_{i=1}^k \sum_{n=a_i}^{b_i} \frac{1}{n}, \quad (1.1)$$

$$a_i < b_i \quad (1 \leq i \leq k), \quad b_i + 1 < a_{i+1} \quad (1 \leq i < k).$$

The separation condition is essential to the question about k : adjacent intervals form a single component, and if repeated or overlapping intervals are allowed the question becomes an easy one about multisets, as Kovač observes in the discussion on the problem's page. For $1 \leq a \leq b$ write $H[a, b] = \sum_{n=a}^b 1/n$. Erdős and Graham reproduce the identity

$$2 = H[2, 7] + H[9, 10] + H[17, 18] + H[34, 35] + H[84, 85], \quad (1.2)$$

found by Montgomery and by Hickerson in response to L.-S. Hahn's Problem E2689 in the *American Mathematical Monthly*.

Theorem A (Erdős Problem 289). *For every sufficiently large integer k , equation (1.1) has a solution with $b_i - a_i \in \{1, 2\}$ for every i .*

Say that intervals $I < I'$ are *separated by at least g integers* if $\min I' - \max I \geq g + 1$; when $g = 1$, they are disjoint and nonadjacent. Families of intervals are counted without ordering their members.

Theorem B (Every positive rational, every large count). *Let $r > 0$ be rational, and let $B \geq 0$ and $g \geq 1$ be integers. There exist constants $A > 0$ and $0 < \delta < 1/5$, and an integer k_0 , depending only on r, B, g , such that the following hold for every integer $k \geq k_0$.*

- (i) *There are at least $2^{k/21}$ distinct families of k intervals $I_1, \dots, I_k$ of integers greater than B , each of length two or three, separated from one another by at least g integers, with*

$$\sum_{i=1}^k \sum_{n \in I_i} \frac{1}{n} = r, \quad \max \bigcup_{i=1}^k I_i \leq Ak^{2-\delta}. \quad (1.3)$$

- (ii) *A family satisfying the same sum, cutoff, separation and denominator bound can be chosen with at most $Ak^{1-\delta/4}$ intervals of length three.*

Theorem A is the case $r = 1$, $B = 0$, $g = 1$ of part (i). Part (ii) does not assert exponential multiplicity among representations with few triples; the constants are effective in principle, and the exponents are not optimised. Results on Egyptian fractions usually prescribe the arithmetic or distribution of individual denominators, as in Croot's short-interval theorem [4], his solution of the colouring conjecture of Erdős and Graham [5], and Bloom's density theorem [2]; see [3] for a survey and [11] for further questions on unit fractions. Here consecutive denominators must occur together while different groups remain separated, and the construction keeps those groups through every arithmetic correction.

Call a finite set S of positive integers *legal* if every maximal run of consecutive elements has length at least two; write $\rho(S) = \sum_{n \in S} 1/n$ and $\kappa(S)$ for the number of maximal runs, its *components*. The problem asks for legal sets of reciprocal sum 1 with every sufficiently large component count. The local operation replaces five triples by six pairs: for $n \geq 2$ the triples $[a, a+2]$ with $a \in \{10n, 15n, 12n+2, 18n+3, 36n+6\}$ lose their first elements and gain the pair $\{6n, 6n+1\}$, which leaves the reciprocal sum unchanged, since $1/10n + 1/15n = 1/6n$ and $1/(12n+2) + 1/(18n+3) + 1/(36n+6) = 1/(6n+1)$, and increases the component count by one. Many such cells are placed so that every choice of their two states remains separated from all other cells and from the arithmetic correction. A *descent* cancels the leading residue at the largest exact prime-power divisor $Q = p^e$ of the remainder's denominator by pairs $\{Qm, Qm+1\}$ with $Qm+1$ factoring into coprime integers below Q ; primitive points on modular hyperbolas supply enough such pairs, restricted sumsets select a batch with any prescribed residue at cost less than $10Q^{-3/2}$, and the levels strictly decrease, so descent above a cutoff L removes less than $E(L) = 10 \sum_{Q>L} Q^{-3/2}$ and leaves a denominator dividing $K = \text{lcm}(1, \dots, L)$. A fixed family of *reserved pairs* $\{Kd+1, Kd+2\}$, grouped into unit-sum sets of indices, absorbs that remainder: extending every pair of one group to a triple adds exactly $1/K$ without changing the component count. All components occupy nine disjoint residue bands modulo a fixed integer R , which is what lets one correction and one completion serve every choice of switches. At scale T a sieve supplies $\asymp T$ cells with no exact prime-power divisor above $T^{1-\varepsilon}$; a prefix of $m_T \asymp T$ of them reaches a prescribed mass with error $O(1/T)$, the correction uses $o(T)$ pairs, and the attainable component counts fill an interval of width m_T ; consecutive scales change m_T by $o(T)$, so these intervals overlap and every large count occurs.

The analytic inputs are the Weil–Estermann bound for Kloosterman sums and Shparlinski's bound for exponential sums with inverse squares, from which the count of primitive points on a modular hyperbola is reproduced uniformly in the unit right-hand side; the additive inputs are the Dias da Silva–Hamidoune and Cauchy–Davenport theorems. Section 2 proves the count-window principle, Sections 3–5 construct the descent, Sections 6–7 the reserve and the cells, Sections 8–9 complete the construction and prove **Theorems A** and **B**, and Appendix A records finite examples and exact enumerations independent of the asymptotic proof.

Two unrefereed proofs of Problem 289 preceded this one on the problem's page. Land (4 September 2026, AI-generated; github.com/beetree/math_erdos_289) claims pairs and triples with all denominators at most $20k$ by an absorption method; Tang (7 September 2026, AI-assisted; github.com/Yuren-Tang/erdos-289) also obtains intervals of length two or three. **Theorem B** differs in scope: every positive rational, prescribed cutoff and separation, exponentially many families

and families with sublinearly many triples; its denominator bound is weaker than the linear bound claimed in the first note. Representation of every rational number in an interval by reciprocal sums of a lacunary sequence, with denominators taken from a prescribed set, is studied by van Doorn and Kovač (arXiv:2509.24971). Whether 1 is a sum of reciprocals over k separated pairs alone for every sufficiently large k remains open: splitting all cells fixes their contribution to the count, and the reserve can still contain triples.

2. INDEPENDENT CHOICES AND COMPONENT COUNTS

An independent switch changes a component count by one. The following proposition turns a family of such switches into representations at every sufficiently large count. Its hypotheses allow the cost of completing the construction to vary arbitrarily between consecutive scales, provided that cost is sublinear. For $\sigma \in \{0, 1\}^m$, let $|\sigma|$ be the number of its coordinates equal to 1.

Proposition 2.1 (Overlapping count windows). *Let $q \geq 1$ be an integer. Suppose that for every sufficiently large integer T there are a set $\mathcal{X}_T$, an integer-valued function κ on $\mathcal{X}_T$, integers $m_T > 0$ and $b_T \geq 0$, and an injection*

$$\Psi_T : \{0, 1\}^{m_T} \longrightarrow \mathcal{X}_T, \quad \kappa(\Psi_T(\sigma)) = qm_T + b_T + |\sigma|.$$

Assume that, as $T \rightarrow \infty$,

$$m_T \asymp T, \quad m_{T+1} - m_T = o(T), \quad b_T = o(T).$$

Then every sufficiently large integer k occurs at some scale T with $k \geq qm_T$, and at least $2^{k/(3(q+2))}$ distinct vectors σ satisfy $\kappa(\Psi_T(\sigma)) = k$.

Proof. Set $a_T = qm_T + b_T$ and restrict the attainable counts to

$$u_T = a_T + \lceil m_T/3 \rceil, \quad v_T = a_T + \lfloor 2m_T/3 \rfloor.$$

Their width satisfies $v_T - u_T \geq m_T/3 - 2 \asymp T$. On the other hand,

$$u_{T+1} - u_T = q(m_{T+1} - m_T) + (b_{T+1} - b_T) + \lceil m_{T+1}/3 \rceil - \lceil m_T/3 \rceil = o(T),$$

because $|b_{T+1} - b_T| \leq b_T + b_{T+1} = o(T)$. The analogous bound holds for $v_{T+1} - v_T$. Hence the integer intervals $[u_T, v_T]$ and $[u_{T+1}, v_{T+1}]$ overlap for all sufficiently large T .

Choose T_* beyond that threshold and large enough that $b_T \leq m_T$ and $m_T \geq 6$ for $T \geq T_*$. Put $k_0 = u_{T_*}$. For $k \geq k_0$, let $T \geq T_*$ be the least index with $v_T \geq k$; it exists since $v_T \rightarrow \infty$. If $T = T_*$, then $u_T \leq k \leq v_T$. Otherwise $v_{T-1} < k \leq v_T$, while overlap gives $u_T \leq v_{T-1}$. Thus again $u_T \leq k \leq v_T$.

Write $s = k - a_T$ and $\ell = \min\{s, m_T - s\}$. Then

$$\frac{m_T}{3} \leq \ell \leq \frac{m_T}{2}, \quad \binom{m_T}{s} = \binom{m_T}{\ell} = \prod_{i=0}^{\ell-1} \frac{m_T - i}{\ell - i} \geq \left(\frac{m_T}{\ell}\right)^\ell \geq 2^{m_T/3}.$$

Every vector of weight s gives count k , and the injection makes the resulting objects distinct. For large T , $b_T \leq m_T$, so

$$qm_T \leq k \leq (q+1)m_T + b_T \leq (q+2)m_T.$$

It follows that $2^{m_T/3} \geq 2^{k/(3(q+2))}$. Increasing k_0 if necessary proves the proposition. $\square$

3. LEADING RESIDUES

Definition 3.1. For a nonzero rational x with reduced denominator D , its *level* is $\lambda(x) = P(D)$; thus $\lambda(x) = 1$ when x is an integer. If $Q = p^e > 1$ is a prime power, define

$$\text{res}_Q(y) = Qy \bmod p \in \mathbb{F}_p \quad \text{for } v_p(y) \geq -e,$$

using the reduction map $\mathbb{Z}_{(p)} \rightarrow \mathbb{F}_p$. When $Q = \lambda(x) > 1$, $\text{res}_Q(x)$ is the *leading residue* of x .

The map res_Q is additive on its domain. If $Q = \lambda(x)$, then Qx is a unit in $\mathbb{Z}_{(p)}$, so the leading residue is nonzero. The following necessary condition explains the role of cancellation.

Lemma 3.2. *Let S be a finite set of positive integers, let p divide an element of S , and let $q = p^e$ be the largest power of p dividing any element of S . If $v_p(\rho(S)) > -e$, then*

$$\sum_{\substack{n \in S \\ q|n}} \left(\frac{n}{q}\right)^{-1} \equiv 0 \pmod{p}.$$

In particular, this condition holds whenever $\rho(S)$ is an integer.

Proof. For $q \mid n$, the quotient n/q is prime to p by the choice of q , and $q/n = (n/q)^{-1}$ in $\mathbb{Z}_{(p)}$. For $q \nmid n$ we have $v_p(n) \leq e - 1$, hence $v_p(q/n) \geq 1$. Therefore

$$q\rho(S) \equiv \sum_{\substack{n \in S \\ q|n}} \left(\frac{n}{q}\right)^{-1} \pmod{p\mathbb{Z}_{(p)}}.$$

The left side is zero modulo p under the stated hypothesis. $\square$

Lemma 3.3. *Let x be a positive rational with $\lambda(x) = Q = p^e > 1$. Suppose that $y \in \mathbb{Q}$ satisfies $v_p(y) \geq -e$ and $\text{res}_Q(y) = \text{res}_Q(x)$, and that every exact prime-power divisor of the reduced denominator of y , other than its p -part, is smaller than Q . Then every exact prime-power divisor of the reduced denominator of $x - y$ is smaller than Q . In particular, $\lambda(x - y) < Q$ if $x \neq y$.*

Proof. The equality of residues gives $Q(x - y) \in p\mathbb{Z}_{(p)}$, so $v_p(x - y) \geq 1 - e$. Thus the p -part of the reduced denominator divides $p^{e-1} < Q$. For a prime $\ell \neq p$,

$$v_\ell(x - y) \geq \min\{v_\ell(x), v_\ell(y)\}.$$

The exact ℓ -parts of both denominators are smaller than Q : for x this follows from $\lambda(x) = Q$, and for y it is a hypothesis. The same is therefore true for $x - y$. $\square$

4. CORRECTION PAIRS

Fix $g \geq 1$ as in [Theorem B](#), and choose an integer t with

$$t \equiv 1 \pmod{35}, \quad 2t - 9 \geq g. \quad (4.1)$$

Then $t \geq 36$. Put $R = 210t$. The centres of the correction pairs will occupy two residue classes modulo R . The other components will occupy seven further classes, arranged in [Section 7](#) so that all distinct components are separated by at least g integers. Constants may depend on the fixed choice of t .

4.1. Points on modular hyperbolas. For an integer $M \geq 2$, a unit u modulo M , and real $U, V \geq 0$, write

$$\begin{aligned} W_{M,u}(U, V) &= \#\{(a, b) \in \mathbb{Z}^2 : 1 \leq a \leq U, 1 \leq b \leq V, ab \equiv u \pmod{M}\}, \\ N_{M,u}(U, V) &= \#\{(a, b) \text{ counted by } W_{M,u}(U, V) : \gcd(a, b) = 1\}. \end{aligned}$$

Set

$$c(M) = \frac{6}{\pi^2} \prod_{\ell \mid M} \left(1 + \frac{1}{\ell}\right)^{-1}, \quad (4.2)$$

where ℓ runs over primes. For $\gcd(d, M) = 1$, let $\bar{d}$ be the inverse of d modulo M .

We use the complete-sum estimates

$$\max_{b \in \mathbb{Z}} \left| \sum_{\substack{1 \leq d \leq M \\ \gcd(d, M) = 1}} e\left(\frac{a\bar{d}^j + bd}{M}\right) \right| \leq (M \gcd(a, M))^{1/2+o(1)}, \quad j \in \{1, 2\}, \quad (4.3)$$

uniformly in the integer a as $M \rightarrow \infty$. For $j = 1$ this is the Weil–Estermann bound [10, Corollary 11.12]. For $j = 2$ it is the special case of [13] recorded in the proof of [14, Lemma 1]. Completion [10, §12.2] gives the same bounds, with an additional factor $O(\log M)$, when d is restricted to an interval of at most M consecutive integers.

Lemma 4.1. *Let $M \geq 2$ and let u be a unit modulo M .*

(a) *For real $0 \leq U, V \leq M$,*

$$W_{M,u}(U, V) = \frac{\varphi(M)UV}{M^2} + O(M^{1/2+o(1)}).$$

(b) *For an interval J of at most M consecutive integers and real $1 \leq X \leq M$,*

$$\begin{aligned} & \#\{d \in J : \gcd(d, M) = 1, (u\bar{d}^2 \bmod M) \in [1, X]\} \\ &= \frac{X}{M} \#\{d \in J : \gcd(d, M) = 1\} + O(M^{1/2+o(1)}). \end{aligned}$$

Both estimates are uniform in all parameters other than M .

Proof. (a) If $U < 1$ or $V < 1$, the count is zero and the main term is $O(1)$. Otherwise, since u is a unit, every solution has $\gcd(a, M) = 1$ and $b \equiv u\bar{a} \pmod{M}$. Orthogonality of additive characters gives

$$W_{M,u}(U, V) = \frac{1}{M} \sum_{h=0}^{M-1} \left(\sum_{1 \leq b \leq V} e\left(\frac{hb}{M}\right) \right) \left(\sum_{\substack{1 \leq a \leq U \\ \gcd(a, M) = 1}} e\left(-\frac{hu\bar{a}}{M}\right) \right).$$

By inclusion and exclusion,

$$\#\{1 \leq a \leq U : \gcd(a, M) = 1\} = \sum_{d|M} \mu(d) \lfloor U/d \rfloor = \frac{\varphi(M)U}{M} + O(\tau(M)).$$

Thus the term $h = 0$ is $\varphi(M)UV/M^2 + O(M^{o(1)})$. For $0 < h < M$, the first inner sum is bounded by $M/\min\{h, M-h\}$, and (4.3), followed by completion, bounds the second by

$$(M \gcd(hu, M))^{1/2+o(1)} = (M \gcd(h, M))^{1/2+o(1)}.$$

The nonzero frequencies therefore contribute at most

$$\begin{aligned} & M^{1/2+o(1)} \sum_{0 < h < M} \frac{\gcd(h, M)^{1/2}}{\min\{h, M-h\}} \\ & \ll M^{1/2+o(1)} \sum_{d|M} d^{1/2} \sum_{\substack{1 \leq h \leq M \\ d|h}} \frac{1}{h} \ll M^{1/2+o(1)} \tau(M) \log M = M^{1/2+o(1)}. \end{aligned}$$

(b) Apply additive characters to the congruence $x \equiv u\bar{d}^2 \pmod{M}$ with $1 \leq x \leq X$ and $d \in J$ prime to M . The zero frequency contributes

$$\frac{\lfloor X \rfloor}{M} \#\{d \in J : \gcd(d, M) = 1\},$$

which differs from the claimed main term by at most 1. At frequency $0 < h < M$, the sum over x is at most $M/\min\{h, M-h\}$, while (4.3) with $j = 2$ and completion bounds the sum over d by $(M \gcd(h, M))^{1/2+o(1)}$. The preceding divisor summation gives the error term. $\square$

Lemma 4.2. *For $M \geq 2$, a unit u modulo M , and real $1 \leq U, V < M$,*

$$N_{M,u}(U, V) = c(M) \frac{UV}{M} + O(M^{3/4+o(1)}),$$

uniformly in u , U and V .

Proof. We give Shparlinski's argument [14, Theorem 3] with a unit right-hand side and fixed truncation parameters. Let M_d count the pairs in $W_{M,u}(U, V)$ for which d divides both coordinates. If $\gcd(d, M) > 1$, then $M_d = 0$, since ab is a unit modulo M . If $\gcd(d, M) = 1$, the substitutions $a = ds$, $b = dt$ give

$$M_d = \#\{(s, t) \in \mathbb{Z}^2 : 1 \leq s \leq U/d, 1 \leq t \leq V/d, st \equiv u\bar{d}^2 \pmod{M}\}.$$

Möbius inversion yields $N_{M,u}(U, V) = \sum_{d \geq 1} \mu(d) M_d$.

For d prime to M , Lemma 4.1(a) gives

$$M_d = \frac{\varphi(M)UV}{d^2 M^2} + O(M^{1/2+o(1)}). \quad (4.4)$$

Also, the product $st \leq UV/d^2$ occupies a fixed residue class modulo M , so it takes at most $UV/(d^2 M) + 1$ values. Each value is less than M^2 and has $M^{o(1)}$ ordered factorizations. Hence

$$M_d \leq \left(\frac{UV}{d^2 M} + 1 \right) M^{o(1)}. \quad (4.5)$$

For the tail, consider integers $d \in [\Delta, 2\Delta)$ with $\Delta \geq M^{1/2}$. If $M_d > 0$, the residue $u\bar{d}^2$ has a representative in $[1, UV/\Delta^2]$. There are no such d when $UV/\Delta^2 < 1$. Otherwise $UV/\Delta^2 < M$, and Lemma 4.1(b) bounds their number by

$$\frac{UV}{\Delta M} + O(M^{1/2+o(1)}).$$

Here only $\Delta < M$ need be considered, since $M_d = 0$ for $d \geq M$. For each eligible d , the product $st < M$ is uniquely determined by its residue and has $M^{o(1)}$ factorizations. Summing over the $O(\log M)$ dyadic intervals beyond $D_2 \geq M^{1/2}$ gives

$$\sum_{d > D_2} M_d \ll \left(\frac{UV}{D_2 M} + M^{1/2} \right) M^{o(1)}. \quad (4.6)$$

Take $D_1 = \lceil M^{1/4} \rceil$ and $D_2 = \lceil M^{1/2} \rceil$. Use (4.4) for $d \leq D_1$, (4.5) for $D_1 < d \leq D_2$, and (4.6) for the remainder. The resulting error is

$$O\left(D_1 M^{1/2+o(1)} + \left(\frac{UV}{D_1 M} + D_2 + \frac{UV}{D_2 M} + M^{1/2} \right) M^{o(1)}\right) = O(M^{3/4+o(1)}),$$

because $UV < M^2$. Extending the main term from $d \leq D_1$ to all d costs at most $O(UV/(MD_1)) = O(M^{3/4})$. Finally,

$$\begin{aligned} \frac{\varphi(M)}{M} \sum_{\substack{d \geq 1 \\ \gcd(d, M)=1}} \frac{\mu(d)}{d^2} &= \prod_{\ell|M} \left(1 - \frac{1}{\ell}\right) \frac{6}{\pi^2} \prod_{\ell \nmid M} \left(1 - \frac{1}{\ell^2}\right)^{-1} \\ &= c(M). \end{aligned}$$

Uniformity in u follows throughout from $\gcd(hu, M) = \gcd(h, M)$. $\square$

By inclusion and exclusion, Lemma 4.1(a) and Lemma 4.2 apply also to boxes $I \times I$, where $I = (A, A'] \cap \mathbb{Z}$ and $0 \leq A < A' < M$, with the same error terms.

4.2. Admissible parameters.

Proposition 4.3. *There is a constant $Q_0 \geq R$ such that every prime power $Q = p^e > Q_0$ admits a set $\mathcal{M}_Q$ of integers satisfying $|\mathcal{M}_Q| \geq Q^{1-o(1)}$ and, for every $m \in \mathcal{M}_Q$,*

$$\begin{aligned} Q/5 < m < Q, \quad p \nmid m, \\ Qm + 1 = ab \quad \text{for some positive integers } a, b < Q \text{ with } \gcd(a, b) = 1, \end{aligned} \quad (4.7)$$

together with

$$Qm \equiv \begin{cases} 42t \pmod{R}, & p \neq 5, \\ 60t \pmod{R}, & p = 5. \end{cases} \quad (4.8)$$

The $o(1)$ tends to zero as $Q \rightarrow \infty$ through prime powers.

Proof. Put $I_Q = (Q/2, Q) \cap \mathbb{Z}$ and $\ell_Q = |I_Q| = Q/2 + O(1)$. We count coprime pairs $(a, b) \in I_Q^2$ satisfying

$$ab \equiv 1 \pmod{Q}, \quad ab \equiv \begin{cases} 42t + 1 \pmod{R}, & p \neq 5, \\ 60t + 1 \pmod{R}, & p = 5. \end{cases} \quad (4.9)$$

Both prescribed classes modulo $R = 2 \cdot 3 \cdot 5 \cdot 7 \cdot t$ are units. They are 1 modulo every prime dividing t . The first is also 1 modulo 2, 3, 7 and 3 modulo 5; the second is 1 modulo 2, 3, 5 and 5 modulo 7, since $t \equiv 1 \pmod{35}$.

The congruences are compatible for $Q > R$. Indeed, if $p \mid R$, then $e > v_p(R)$. For $p \neq 5$,

$$v_p(42t) = v_p(210t) = v_p(R);$$

for $p = 5$, we have $v_5(R) = v_5(60t) = 1$ because $5 \nmid t$. Thus both congruences prescribe 1 modulo $\gcd(Q, R)$. They combine to $ab \equiv u \pmod{M}$ for a unit u and

$$M = \text{lcm}(Q, R), \quad Q < M \leq RQ.$$

Every prime divisor of M divides Rp , so $c(M) \geq c_0 > 0$ with c_0 depending only on t . Applying Lemma 4.2 to I_Q^2 gives

$$\begin{aligned} \#\{(a, b) \in I_Q^2 : \gcd(a, b) = 1, (4.9)\} &= c(M) \frac{\ell_Q^2}{M} + O(Q^{3/4+o(1)}), \\ c(M) \frac{\ell_Q^2}{M} &\geq \frac{c_0}{5R} Q \quad (Q \text{ sufficiently large}). \end{aligned} \quad (4.10)$$

The box lies below M , and the second line uses $\ell_Q \geq Q/2 - 1$ and $M \leq RQ$.

Discard the pairs with $v_p(ab - 1) > e$. If $e = 1$, there are none: $0 < ab - 1 < p^2$. If $e \geq 2$, a discarded pair satisfies $ab \equiv 1 \pmod{pQ}$ as well as the congruence modulo R . These again combine to a unit class, now modulo $\text{lcm}(pQ, R) = pM$. Counting without the coprimality restriction and using Lemma 4.1(a), the number discarded is at most

$$\frac{\varphi(pM)\ell_Q^2}{(pM)^2} + O((pM)^{1/2+o(1)}) = \frac{\varphi(M)\ell_Q^2}{pM^2} + O(Q^{3/4+o(1)}). \quad (4.11)$$

Here $\varphi(pM) = p\varphi(M)$ and $pM \leq RQ^{3/2}$. The ratio of its main term to that in (4.10) is

$$\frac{1}{p} \frac{\varphi(M)/M}{c(M)} = \frac{\pi^2}{6p} \prod_{\ell \mid M} \left(1 - \frac{1}{\ell^2}\right) \leq \frac{\pi^2}{12} < 1.$$

It follows that at least $c_1 Q$ coprime pairs remain, for some fixed $c_1 > 0$, with $v_p(ab - 1) = e$.

For each remaining pair, set $m = (ab - 1)/Q$. Then $p \nmid m$,

$$Q/5 \leq Q/4 - 1/Q < m < Q \quad (Q \geq 5),$$

and reduction modulo R gives (4.8). A fixed m comes from at most $\tau(Qm+1) = Q^{o(1)}$ ordered factorizations, since $Qm+1 < Q^2$. Thus the set $\mathcal{M}_Q$ of resulting parameters has at least $c_1 Q^{1-o(1)} = Q^{1-o(1)}$ elements. $\square$

Remark 4.4. The box just used has area $\ell_Q^2 \asymp Q^2 \asymp M^2$, with constants depending on R . Its main term is $\asymp M$ and dominates $O(M^{3/4+o(1)})$; the sharper area-dependent estimate in [14] is not required. Coprimality of a, b ensures that every prime power dividing $Qm+1 = ab$ is smaller than Q . The discarded pairs need not be coprime, which is why the simpler count in Lemma 4.1(a) suffices for them. General unit right-hand sides are also discussed in [14, §4] and [15].

4.3. Correction pairs.

Definition 4.5. For $Q = p^e > Q_0$ and $m \in \mathcal{M}_Q$, the *correction pair* with level Q and parameter m is

$$\mathcal{P}_{Q,m} = \{Qm, Qm+1\}.$$

Its *centre* is the even integer Qm .

Lemma 4.6. Let $Q = p^e > Q_0$ and $m \in \mathcal{M}_Q$.

- (a) $P(Qm) = Q$ and $P(Qm+1) < Q$.
- (b) $v_p(\rho(\mathcal{P}_{Q,m})) = -e$ and $\text{res}_Q(\rho(\mathcal{P}_{Q,m})) = m^{-1} \pmod p$.
- (c) $\rho(\mathcal{P}_{Q,m}) < 10/Q^2$.
- (d) Every element of $\mathcal{P}_{Q,m}$ exceeds $Q^2/5$ and is congruent to one of $42t, 42t+1, 60t, 60t+1$ modulo R .
- (e) Distinct correction pairs, including pairs at different levels, are disjoint and have centres at distance at least $18t$. They are therefore separated by at least $18t-2 \geq g$ integers.

Proof. (a) The exact p -part of Qm is Q , since $p \nmid m$; every other exact prime-power factor divides $m < Q$. In the factorization $Qm+1 = ab$, the factors $a, b < Q$ are coprime, so each prime power dividing their product divides one of them.

(b) The two summands $1/(Qm)$ and $1/(Qm+1)$ have respective p -adic valuations $-e$ and 0 . Their sum therefore has valuation $-e$, and

$$Q\rho(\mathcal{P}_{Q,m}) = \frac{1}{m} + \frac{Q}{Qm+1} \equiv m^{-1} \pmod{p\mathbb{Z}_{(p)}}.$$

(c) Since $m > Q/5$, $\rho(\mathcal{P}_{Q,m}) < 2/(Qm) < 10/Q^2$.

(d) This follows from (4.8) and $Qm > Q^2/5$.

(e) Centres are even, so distinct pairs can intersect only if their centres coincide. By (a), a centre determines $Q = P(Qm)$ and then m . Distinct centres are congruent to $42t$ or $60t$ modulo $210t$; their difference is therefore at least $18t$ in absolute value. This leaves at least $18t-2 \geq 2t-9 \geq g$ integers between the pairs. $\square$

4.4. Every residue at small cost. For $A \subseteq \mathbb{F}_p$ and $h \geq 1$, let $h^\wedge A$ be the set of sums of h distinct elements of A . The theorem of Dias da Silva and Hamidoune [7, 1] gives

$$|h^\wedge A| \geq \min\{p, h(|A| - h) + 1\}. \quad (4.12)$$

We also use Cauchy–Davenport [6]: $|A+B| \geq \min\{p, |A|+|B|-1\}$ for nonempty subsets $A, B \subseteq \mathbb{F}_p$. For proofs of both results see [12, Chapters 2 and 3].

For $Q = p^e > Q_0$, put

$$N(Q) = \begin{cases} \lceil 2Q/|\mathcal{M}_Q| \rceil, & e = 1, \\ p-1, & e \geq 2. \end{cases} \quad (4.13)$$

By Proposition 4.3, $N(Q) \leq Q^{o(1)}$ at prime levels; at proper prime-power levels, $N(Q) \leq Q^{1/2}$.

Proposition 4.7. *There is a constant $Q_1 \geq Q_0$ such that, for every prime power $Q = p^e > Q_1$ and every $c \in \mathbb{F}_p$, some $\mathcal{S} \subseteq \mathcal{M}_Q$ satisfies*

$$\sum_{m \in \mathcal{S}} m^{-1} \equiv c \pmod{p}, \quad |\mathcal{S}| \leq N(Q) \leq Q^{1/2}, \quad \sum_{m \in \mathcal{S}} \rho(\mathcal{P}_{Q,m}) < 10Q^{-3/2}.$$

Proof. If $e = 1$, distinct parameters in $\mathcal{M}_p \subset (0, p)$ have distinct inverses modulo p . Hence

$$A = \{m^{-1} \bmod p : m \in \mathcal{M}_p\} \quad \text{has size} \quad s = |\mathcal{M}_p| \geq p^{1-o(1)}.$$

For sufficiently large p , the integer $h = N(p) = \lceil 2p/s \rceil$ satisfies $h \leq s/2$, and thus $h(s-h) \geq hs/2 \geq p$. Equation (4.12) gives $h^\wedge A = \mathbb{F}_p$. Every residue is therefore obtained from exactly h distinct parameters.

If $e \geq 2$, then $p \leq Q^{1/2}$ and $|\mathcal{M}_Q| \geq Q^{1-o(1)} \geq p-1$ for large Q . Choose $p-1$ distinct parameters and denote their nonzero inverse residues by $c_1, \dots, c_{p-1}$; the residues themselves may repeat. Repeated Cauchy–Davenport gives

$$|\{0, c_1\} + \dots + \{0, c_i\}| \geq \min\{p, i+1\}.$$

For $i = p-1$ all residues occur. The corresponding subsets use at most $p-1 = N(Q)$ distinct parameters.

In the first case $N(Q) \leq Q^{o(1)}$, and in the second $N(Q) = p-1 < Q^{1/2}$. Increasing Q_1 therefore ensures $|\mathcal{S}| \leq N(Q) \leq Q^{1/2}$ in both cases. By Lemma 4.6(c), their total reciprocal mass is less than $10|\mathcal{S}|/Q^2 \leq 10Q^{-3/2}$; if $\mathcal{S}$ is empty, the claimed bound holds directly. $\square$

Corollary 4.8. *For an integer $L \geq Q_1$ and real $Y \geq L$, put*

$$E(L) = 10 \sum_{Q > L} Q^{-3/2}, \quad H(Y) = \sum_{Q_1 < Q \leq Y} N(Q),$$

where both sums run over prime powers. Then $E(L) \leq 20L^{-1/2}$ and, for every $\nu > 0$,

$$H(Y) \ll_\nu Y^{1+\nu}.$$

Proof. Comparison with the sum over all integers gives

$$E(L) \leq 10 \sum_{n > L} n^{-3/2} \leq 10 \int_L^\infty s^{-3/2} ds = 20L^{-1/2}.$$

For prime levels, $N(p) \leq p^\nu$ once p exceeds a fixed threshold, so their contribution to $H(Y)$ is $O_\nu(1) + O(Y^{1+\nu})$. The proper prime powers contribute at most

$$\begin{aligned} \sum_{e \geq 2} \sum_{p^e \leq Y} (p-1) &\leq \sum_{n \leq Y^{1/2}} n + \sum_{3 \leq e \leq \log_2 Y} \sum_{n \leq Y^{1/e}} n \\ &= O(Y) + O(Y^{2/3} \log Y). \end{aligned}$$

Combining the two estimates proves the claim. $\square$

5. DESCENT

The corrections are applied in decreasing order of level; their total reciprocal mass is summable, independently of the initial denominator.

Theorem 5.1. *Let $L \geq Q_1$ be an integer, put $K = \text{lcm}(1, \dots, L)$, and let $Y \geq L$. If x is a positive rational with $\lambda(x) \leq Y$ and $x > E(L)$, there is a finite set $\mathcal{C}$ of correction pairs with the following properties.*

- (a) *Each pair has a level Q with $L < Q \leq Y$. At most $N(Q)$ pairs occur at each level, and $|\mathcal{C}| \leq H(Y)$. The construction processes the levels in strictly decreasing order.*

- (b) Any two pairs are separated by at least g integers. Every denominator exceeds $L^2/5$ and is congruent to one of $42t$, $42t + 1$, $60t$, $60t + 1$ modulo R .
- (c) The remainder

$$x' = x - \sum_{\mathcal{P} \in \mathcal{C}} \rho(\mathcal{P})$$

satisfies $x - E(L) < x' \leq x$ and $Kx' \in \mathbb{Z}$.

Proof. Start with $x_0 = x$. Suppose that corrections have been chosen at distinct levels $\Lambda_0 > \dots > \Lambda_{i-1} > L$, leaving a positive remainder x_i . If $\lambda(x_i) \leq L$, stop. Otherwise write

$$\Lambda_i = \lambda(x_i) = p_i^{e_i}, \quad c_i = \text{res}_{\Lambda_i}(x_i).$$

By [Proposition 4.7](#), choose $\mathcal{S}_i \subseteq \mathcal{M}_{\Lambda_i}$ such that

$$\sum_{m \in \mathcal{S}_i} m^{-1} \equiv c_i \pmod{p_i}, \quad |\mathcal{S}_i| \leq N(\Lambda_i),$$

and set

$$y_i = \sum_{m \in \mathcal{S}_i} \rho(\mathcal{P}_{\Lambda_i, m}), \quad x_{i+1} = x_i - y_i.$$

The cost of this batch is $y_i < 10\Lambda_i^{-3/2}$. Including the current level, all levels processed so far are distinct prime powers greater than L . Consequently

$$\sum_{j=0}^i y_j < 10 \sum_{Q > L} Q^{-3/2} = E(L), \quad x_{i+1} > x - E(L) > 0.$$

By [Lemma 4.6\(a\),\(b\)](#), y_i has valuation at least $-e_i$ at p_i , has residue c_i at level Λ_i , and has no other exact prime-power denominator factor as large as Λ_i . Indeed, every such factor divides a denominator of one of its summands. Thus [Lemma 3.3](#) gives $\lambda(x_{i+1}) < \Lambda_i$. This proves the induction, including strict decrease of the processed levels. Since positive integers cannot decrease indefinitely, the construction stops.

Let x' be the final remainder and let $\mathcal{C}$ consist of all pairs chosen. Then $x - E(L) < x' \leq x$. At termination every exact prime-power divisor of the denominator of x' is at most L , and hence divides K ; therefore $Kx' \in \mathbb{Z}$. There are at most $N(Q)$ pairs at each processed level, so

$$|\mathcal{C}| = \sum_i |\mathcal{S}_i| \leq \sum_i N(\Lambda_i) \leq H(Y).$$

Finally, [Lemma 4.6\(d\),\(e\)](#) supplies separation, the residue classes, and the lower bound on every denominator. $\square$

6. RESERVED PAIRS

The descent reduces the remaining denominator to a divisor of $K = \text{lcm}(1, \dots, L)$. A finite family of pairs, reserved in advance, absorbs every remainder in the required range. Completing a pair $\{Kd + 1, Kd + 2\}$ by adjoining Kd adds $1/(Kd)$ to its reciprocal sum without changing its number of components. The reservoir is fixed before any cells or correction pairs are chosen.

Lemma 6.1. *For every integer $M \geq 1$ there is a finite set $\mathcal{D}$ of integers greater than M such that $\sum_{d \in \mathcal{D}} 1/d = 1$.*

Proof. Let $N > M$ be the least integer for which $\sum_{d=M+1}^N 1/d \geq 1$. Such an N exists by divergence of the harmonic series. In the equality case take $\mathcal{D} = \{M + 1, \dots, N\}$. Otherwise put

$$x = 1 - \sum_{d=M+1}^{N-1} \frac{1}{d}.$$

Minimality of N and the strict inequality at N give $0 < x < 1/N$.

Apply the greedy unit-fraction algorithm to x . At a nonzero stage, write the remainder as a/b in lowest terms and put $q = \lceil b/a \rceil$. Then

$$\frac{a}{b} - \frac{1}{q} = \frac{aq - b}{bq}, \quad 0 \leq aq - b < a.$$

If the remainder is nonzero, reduction to lowest terms therefore strictly decreases its numerator. The process terminates. Moreover,

$$0 < \frac{a}{b} - \frac{1}{q} < \frac{a}{bq} < \frac{1}{q(q-1)},$$

so the next denominator is greater than $q(q-1) \geq q$. The first denominator is $\lceil 1/x \rceil > N$, and all subsequent denominators are distinct and larger. Adjoin these denominators to $\{M+1, \dots, N-1\}$ to obtain $\mathcal{D}$. $\square$

6.1. The fixed constants. Fix $r > 0$ and $B \geq 0$ as in Theorem B; the integers g, t, R remain as fixed in Section 4. Set

$$\eta = \frac{r}{1000}. \quad (6.1)$$

By Corollary 4.8, we may choose an integer L satisfying

$$L \geq Q_1, \quad E(L) < \frac{\eta}{4}, \quad L \geq R, \quad L \geq \frac{10}{\eta}, \quad L^2 > 5B. \quad (6.2)$$

Define

$$K = \text{lcm}(1, \dots, L), \quad J = \lceil 4K\eta \rceil. \quad (6.3)$$

Thus $R \mid K$ and $K\eta \geq L\eta \geq 10$.

Apply Lemma 6.1 successively J times. In the first application use $M = \max\{1, B\}$, and thereafter take M to be the maximum of 1, B and all indices already chosen. This gives pairwise disjoint finite sets $\mathcal{D}_0, \dots, \mathcal{D}_{J-1}$, all of whose elements exceed B , with

$$\sum_{d \in \mathcal{D}_j} \frac{1}{d} = 1 \quad (0 \leq j < J). \quad (6.4)$$

Put

$$U = \bigcup_{j=0}^{J-1} \mathcal{D}_j, \quad \mathcal{A} = \bigcup_{d \in U} \{Kd+1, Kd+2\}, \quad C_0 = |U|, \quad A_0 = \rho(\mathcal{A}). \quad (6.5)$$

For $d \in U$, write $R_d = \{Kd+1, Kd+2\}$ and call R_d a *reserved pair*. The constants and sets just chosen remain fixed throughout the construction.

Lemma 6.2. (a) *The components of $\mathcal{A}$ are the C_0 reserved pairs R_d . They are separated by at least g integers, all their elements exceed B , and $A_0 < 10\eta$.*

(b) *If $0 < x \leq 2\eta$ and $Kx \in \mathbb{Z}$, there is a subset $\mathcal{D} \subseteq U$ such that*

$$\sum_{d \in \mathcal{D}} \frac{1}{Kd} = x.$$

For every subset $\mathcal{D} \subseteq U$, the set $\mathcal{A}' = \mathcal{A} \cup \{Kd : d \in \mathcal{D}\}$ is legal and satisfies

$$\rho(\mathcal{A}') = A_0 + \sum_{d \in \mathcal{D}} \frac{1}{Kd}, \quad \kappa(\mathcal{A}') = C_0.$$

Its components are the pairs R_d for $d \in U \setminus \mathcal{D}$ and the triples $\{Kd, Kd+1, Kd+2\}$ for $d \in \mathcal{D}$. Any two are separated by at least g integers, and every element of $\mathcal{A}'$ is congruent to 0, 1 or 2 modulo R .

Proof. Distinct indices in U differ by at least one, so even the enlarged intervals $[Kd, Kd + 2]$ have at least $K - 3$ integers between them. Since $K \geq R = 210t$ and $2t - 9 \geq g$, this gives the required separation for both $\mathcal{A}$ and $\mathcal{A}'$. Their stated pairs and triples are therefore exactly their components. The indices in U exceed B , so all these components lie above B .

Using (6.4) and $K\eta \geq 10$, we have

$$A_0 < \frac{2}{K} \sum_{d \in U} \frac{1}{d} = \frac{2J}{K} \leq 8\eta + \frac{2}{K} \leq \left(8 + \frac{1}{5}\right)\eta < 10\eta.$$

This proves (a).

For the existence assertion in (b), write $w = Kx$. Then w is an integer with $1 \leq w \leq 2K\eta < J$. Take $\mathcal{D} = \mathcal{D}_0 \cup \dots \cup \mathcal{D}_{w-1}$. Equation (6.4) gives

$$\sum_{d \in \mathcal{D}} \frac{1}{Kd} = \frac{1}{K} \sum_{j=0}^{w-1} \sum_{d \in \mathcal{D}_j} \frac{1}{d} = \frac{w}{K} = x.$$

For an arbitrary subset $\mathcal{D} \subseteq U$, each adjoined integer Kd is new, extends precisely its reserved pair, and contributes $1/(Kd)$. This proves both formulas. Finally $R \mid K$, so the enlarged components occupy only the residues 0, 1 and 2 modulo R . $\square$

7. CELLS

Two configurations with the same reciprocal sum and component counts differing by one are placed so that every choice between them coexists with the reserved and correction pairs.

Definition 7.1. For an integer $d \geq 1$, put $n = Kd + 8t$ and

$$\mathcal{A}_n = \{10n, 15n, 12n + 2, 18n + 3, 36n + 6\}.$$

The *cell* of index d is

$$Z_d = \bigcup_{a \in \mathcal{A}_n} [a, a + 2],$$

and its *split state* is

$$Z_d^* = [6n, 6n + 1] \cup \bigcup_{a \in \mathcal{A}_n} [a + 1, a + 2].$$

Write $h(d)$ for their common reciprocal sum, whose equality is proved below.

Thus splitting deletes the first element of each of five triples and inserts one pair. The fifteen elements of Z_d are the values $\alpha d + \beta$ of the forms

$$\begin{aligned} (\alpha, \beta) = & (10K, 80t + i), \quad (15K, 120t + i), \quad (12K, 96t + 2 + i), \\ & (18K, 144t + 3 + i), \quad (36K, 288t + 6 + i), \quad i \in \{0, 1, 2\}. \end{aligned} \tag{7.1}$$

The only elements of Z_d^* outside Z_d are $6Kd + 48t$ and $6Kd + 48t + 1$.

Lemma 7.2. For every $d \geq 1$:

- (a) $\rho(Z_d^*) = \rho(Z_d) = h(d)$;
- (b) Z_d has five components, each of length three, and Z_d^* has six components, each of length two;
- (c)

$$\frac{1}{K(d+1)} \leq h(d) \leq \frac{1}{Kd}, \quad h(d) = \frac{1}{Kd} + O_{K,t}(d^{-2}). \tag{7.2}$$

In particular both states are legal.

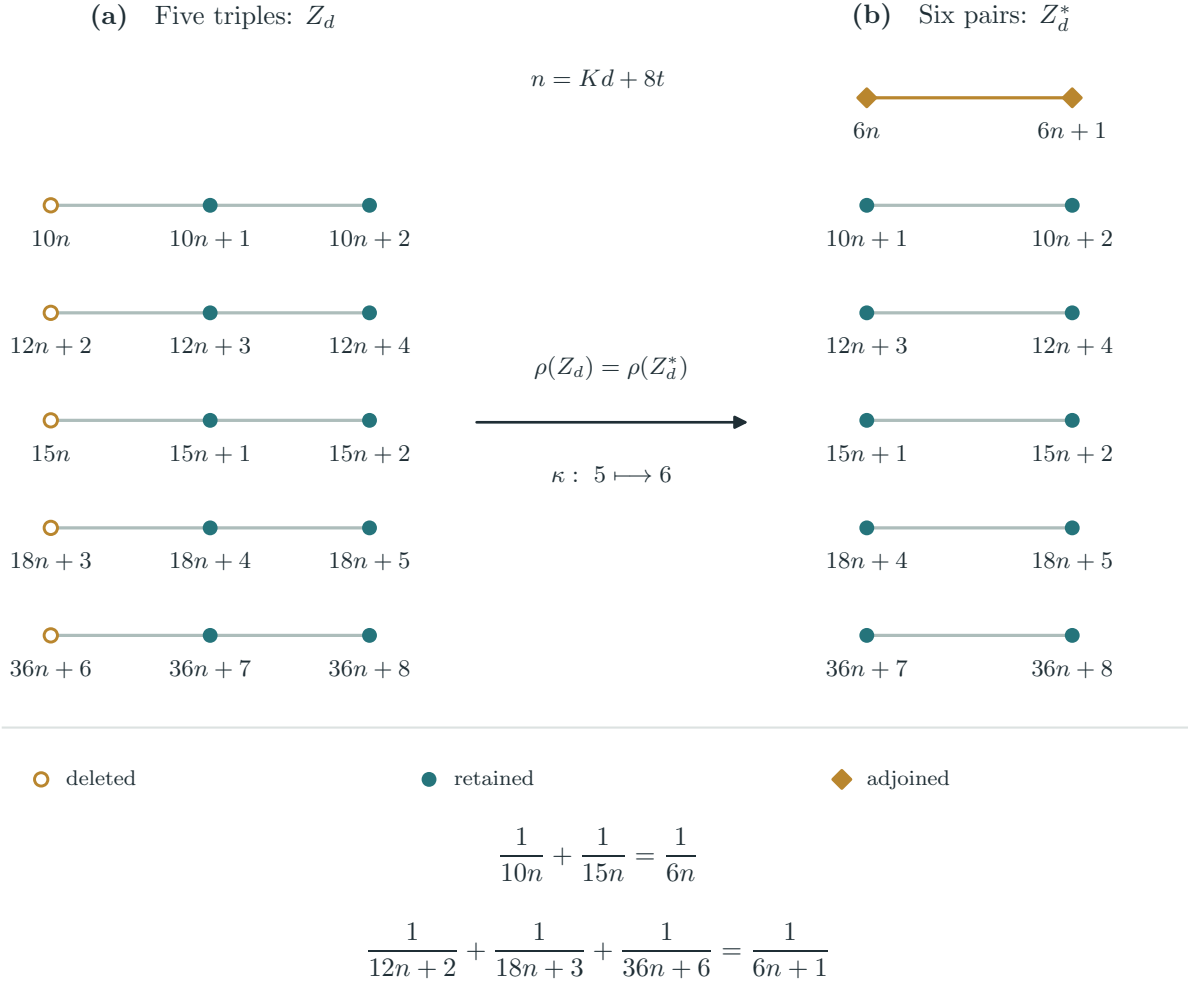


FIGURE 1. The two states of one cell. The ten retained denominators are marked by teal circles. The five open amber circles are removed, and the two amber diamonds are inserted; the two displayed identities show that the exchanged reciprocal masses agree. The five triples become six pairs, so the reciprocal sum is fixed while the component count increases by one. Positions are schematic.

Proof. The deleted and inserted elements have the same reciprocal sum, since

$$\frac{1}{10n} + \frac{1}{15n} = \frac{1}{6n}$$

and

$$\frac{1}{12n+2} + \frac{1}{18n+3} + \frac{1}{36n+6} = \frac{1}{6n+1} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} \right) = \frac{1}{6n+1}.$$

All other elements are retained. This proves (a).

The starting points of the triples, in increasing order, are

$$10n, \quad 12n+2, \quad 15n, \quad 18n+3, \quad 36n+6,$$

and those of the split pairs are

$$6n, \quad 10n+1, \quad 12n+3, \quad 15n+1, \quad 18n+4, \quad 36n+7.$$

In either list consecutive starting points differ by at least $2n$. The displayed intervals are therefore disjoint and nonadjacent, proving (b).

For every form in (7.1), we have $0 < \beta < \alpha$: indeed $\beta \leq 288t + 8$ whereas $\alpha \geq 10K \geq 2100t$. Thus

$$\frac{1}{\alpha(d+1)} \leq \frac{1}{\alpha d + \beta} \leq \frac{1}{\alpha d}.$$

The sum of the reciprocals of the fifteen coefficients is

$$\frac{3}{K} \left(\frac{1}{10} + \frac{1}{15} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36} \right) = \frac{1}{K}.$$

Summing the preceding bounds proves the inequalities in (c), and expanding each fixed form as $1/(\alpha d + \beta) = 1/(\alpha d) + O_{K,t}(d^{-2})$ proves the asymptotic formula. $\square$

The residue classes of all elements in either state are independent of d . Modulo $R = 210t$, they lie in the six bands

$$[48t, 48t+1], [78t+6, 78t+8], [80t, 80t+2], [96t+2, 96t+4], [120t, 120t+2], [144t+3, 144t+5]. \quad (7.3)$$

Here the first band contains the inserted pair; the second comes from $36n + 6 \equiv 78t + 6 \pmod{R}$. The remaining bands come from the triples beginning at $10n$, $12n + 2$, $15n$ and $18n + 3$, respectively.

The reserved components occupy $[0, 2]$, by Lemma 6.2(b), and the correction pairs occupy $[42t, 42t + 1]$ or $[60t, 60t + 1]$, by Lemma 4.6(d). All nine bands, in cyclic order, are

$$\begin{aligned} &[0, 2], [42t, 42t + 1], [48t, 48t + 1], [60t, 60t + 1], [78t + 6, 78t + 8], \\ &[80t, 80t + 2], [96t + 2, 96t + 4], [120t, 120t + 2], [144t + 3, 144t + 5]. \end{aligned} \quad (7.4)$$

The numbers of residues strictly between successive bands, including the final gap back to 0, are

$$42t - 3, \quad 6t - 2, \quad 12t - 2, \quad 18t + 4, \quad 2t - 9, \quad 16t - 1, \quad 24t - 5, \quad 24t, \quad 66t - 6.$$

Each is at least $2t - 9 \geq g$.

Proposition 7.3. *Let $\mathcal{A}'$ be a completed reserved set as in Lemma 6.2(b), let $\mathcal{C}$ be a finite set of correction pairs, and let E be a finite set of positive integers. Choose $Z'_d \in \{Z_d, Z_d^*\}$ for each $d \in E$. Then*

$$S = \mathcal{A}' \cup \bigcup_{\mathcal{P} \in \mathcal{C}} \mathcal{P} \cup \bigcup_{d \in E} Z'_d$$

is legal. Its components are exactly the components of $\mathcal{A}'$, the pairs in $\mathcal{C}$, and the components of the chosen cell states. They have length two or three and are separated by at least g integers. Moreover,

$$\rho(S) = \rho(\mathcal{A}') + \sum_{\mathcal{P} \in \mathcal{C}} \rho(\mathcal{P}) + \sum_{d \in E} h(d), \quad \kappa(S) = C_0 + |\mathcal{C}| + \sum_{d \in E} \kappa(Z'_d).$$

Proof. Each proposed component has length two or three and lies, modulo R , within a single band of (7.4). We verify that any two proposed components are disjoint and separated by the required gap.

Components in different bands have disjoint residues, so they cannot intersect. Order such components as $X < Y$. The positive difference $\min Y - \max X$ is congruent to a forward difference between their bands on the residue circle. That difference crosses at least one of the gaps listed above. Consequently $\min Y - \max X \geq 2t - 8 \geq g + 1$.

Components in the same band belong to the same kind of construction. For reserved components, distinct indices give a difference of at least $K - 2$ from the end of the earlier component to the beginning of the later one. For cell components, each band specifies one of the six linear families. Different indices in that family give a difference of at least $6K - 2$ between the respective endpoints; the possible change from a triple to its final pair does not decrease this bound. For correction pairs, distinct centres differ by at least $18t$ by Lemma 4.6(e), so the corresponding endpoint difference

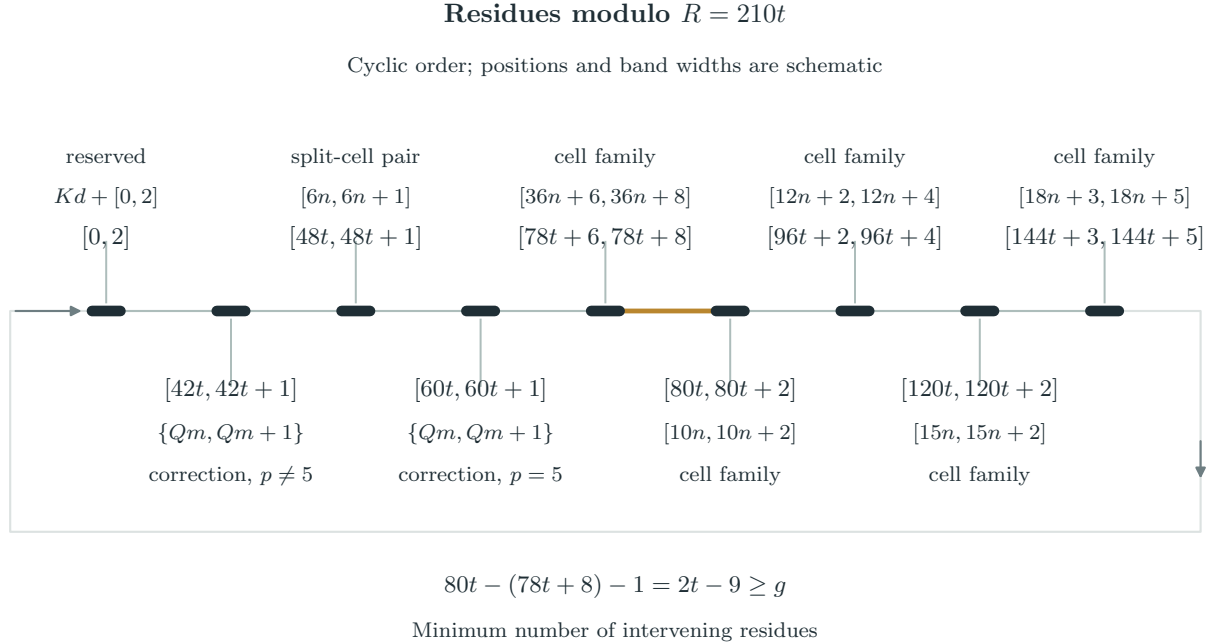


FIGURE 2. The nine residue bands modulo $R = 210t$, in cyclic order. Reserved components and their completions, correction pairs, and both cell states occupy separate bands. The smallest gap contains $2t - 9 \geq g$ unused residues; the remaining gaps are larger. Within each band, different components are separated by the spacing estimates in Proposition 7.3. Band widths and intervening distances are schematic.

is at least $18t - 1$. These bounds establish disjointness in each case and leave at least g integers between the components.

Thus no proposed components merge, and none of their elements are repeated. The component count is additive, as is the reciprocal sum. The latter uses $\rho(Z'_d) = h(d)$ from Lemma 7.2(a). $\square$

Corollary 7.4. *In the setting of Proposition 7.3, suppose $Z'_d = Z_d$ for the indices in a subset $E' \subseteq E$. Replacing these states by Z_d^* preserves the reciprocal sum, legality and separation, and increases the component count by exactly $|E'|$.*

Proof. Apply Proposition 7.3 before and after the replacements, and use Lemma 7.2(a),(b). $\square$

8. POWERSMOOTH CELLS AT A FIXED SCALE

A family of cells is selected whose reciprocal sum leaves a small positive remainder; powersmoothness bounds the number of correction pairs needed to descend that remainder, the reservoir completes it exactly, and the same completion remains valid after any collection of cells is split.

Set

$$G = r - A_0 - 2\eta > 0, \quad (8.1)$$

where positivity follows from $A_0 < 10\eta$ and $\eta = r/1000$. Choose an integer $C > 1$ such that

$$\frac{1}{K} \log C > 2r, \quad (8.2)$$

and then choose ε with

$$0 < \varepsilon < \min \left\{ \frac{1}{10}, \frac{rK}{100(C-1)} \right\}. \quad (8.3)$$

Put $\Gamma = 36KC + 300t$. All three constants are fixed. For each positive integer T , write

$$Y_T = T^{1-\varepsilon}. \quad (8.4)$$

Definition 8.1. A cell Z_d is *admissible at scale T* if all fifteen of its elements are Y_T -powersmooth. Let

$$\mathcal{L}_T = \{d \in [T, CT] \cap \mathbb{Z} : Z_d \text{ is admissible at scale } T\}.$$

Only the unsplit cell is required to be powersmooth. The two elements inserted in its split state need no such restriction, since splitting preserves the reciprocal sum exactly.

8.1. Few cells are lost.

Lemma 8.2. *As $T \rightarrow \infty$,*

$$\#([T, CT] \cap \mathbb{Z}) \setminus \mathcal{L}_T \leq 30(C-1)\varepsilon T + o(T),$$

where the error depends on K, t, C and ε .

Proof. If $d \in [T, CT] \cap \mathbb{Z}$ is inadmissible, one of the fifteen forms $\alpha d + \beta$ in (7.1) is divisible by a prime power $q > Y_T$. Every such divisor satisfies

$$q \leq \alpha d + \beta \leq 36KCT + 288t + 8 < \Gamma T.$$

We bound the number of indices excluded by each form, treating primes and proper prime powers separately.

First let $q = p$ be prime. The coefficient α is one of $10K, 12K, 15K, 18K$ and $36K$. Its prime divisors are therefore at most L , since $L \geq R$. For sufficiently large T , therefore, $p > Y_T > L$ implies $\gcd(p, \alpha) = 1$. The congruence $\alpha d + \beta \equiv 0 \pmod{p}$ selects one residue class, containing at most $(C-1)T/p + 1$ indices in $[T, CT]$. Mertens' theorem [9, Theorem 427] gives

$$\begin{aligned} \sum_{Y_T < p < \Gamma T} \frac{1}{p} &= \log \frac{\log(\Gamma T)}{\log Y_T} + o(1) \\ &= \log \frac{\log \Gamma + \log T}{(1-\varepsilon) \log T} + o(1) = -\log(1-\varepsilon) + o(1) \leq 2\varepsilon + o(1). \end{aligned}$$

Also $\pi(\Gamma T) = o(T)$ by Chebyshev's bound [9, Theorem 7], since Γ is fixed. Summing the residue-class bounds over p excludes at most

$$(C-1)T \sum_{Y_T < p < \Gamma T} \frac{1}{p} + \pi(\Gamma T) \leq 2(C-1)\varepsilon T + o(T)$$

indices for this form.

Now let $q = p^a$ with $a \geq 2$, and put $\gamma = \gcd(\alpha, q) \leq \alpha \leq 36K$. If $\alpha d + \beta \equiv 0 \pmod{q}$ is soluble, its solutions constitute one residue class modulo q/γ . Their number is at most $\gamma((C-1)T/q + 1)$. The total contribution of proper prime powers is therefore at most

$$36K \left((C-1)T \sum_{\substack{q > Y_T \\ q \text{ proper}}} \frac{1}{q} + \#\{q < \Gamma T : q \text{ is a proper prime power}\} \right).$$

The series of reciprocals of proper prime powers converges, so the first term is $o(T)$. The number of proper prime powers below x is at most

$$\sum_{2 \leq a \leq \log_2 x} x^{1/a} \leq x^{1/2} + O(x^{1/3} \log x) = O(x^{1/2}),$$

so the second term is also $o(T)$. A union bound over the fifteen forms proves the assertion. $\square$

8.2. Selection of a prefix.

Lemma 8.3. *For every sufficiently large integer T :*

$$(a) \sum_{d \in \mathcal{L}_T} h(d) > r;$$

(b) *there is a least integer $D_T \in [T, CT]$ such that $\sum_{\substack{d \in \mathcal{L}_T \\ d \leq D_T}} h(d) \geq G$. Writing*

$$\mathcal{E}_T = \{d \in \mathcal{L}_T : d \leq D_T\}, \quad m_T = |\mathcal{E}_T|,$$

we have

$$G \leq \sum_{d \in \mathcal{E}_T} h(d) < G + \frac{1}{KT} \quad (8.5)$$

and

$$KGT \leq m_T \leq (C-1)T + 1. \quad (8.6)$$

Proof. The asymptotic formula in (7.2) gives

$$\sum_{T \leq d \leq CT} h(d) = \frac{1}{K} \sum_{T \leq d \leq CT} \frac{1}{d} + O_{K,t}(T^{-1}) = \frac{1}{K} \log C + o(1) > 2r$$

for sufficiently large T . Each omitted term is at most $1/(KT)$. By Lemma 8.2, the omitted mass is consequently at most

$$\frac{30(C-1)\varepsilon}{K} + o(1) < \frac{3r}{10} + o(1) < \frac{r}{2}$$

for large T . The remaining mass exceeds r , proving (a).

Since $G < r$, the stopping index D_T exists. It belongs to $\mathcal{L}_T$, and the partial sum immediately before its contribution is less than G . Adding $h(D_T) \leq 1/(KT)$ proves (8.5). The same upper bound on individual terms gives $G \leq m_T/(KT)$. Finally $\mathcal{E}_T \subseteq [T, CT] \cap \mathbb{Z}$, giving the upper bound in (8.6). $\square$

8.3. One correction for all cell states. For $\sigma \in \{0, 1\}^{\mathcal{E}_T}$, let $|\sigma|$ denote the number of coordinates equal to one, and write $Z_d^{(0)} = Z_d$ and $Z_d^{(1)} = Z_d^*$.

Theorem 8.4. *There is a constant A_1 , depending only on r, B and g , with the following property. For every sufficiently large integer T , there are a completed reserved set $\mathcal{A}'_T$ as in Lemma 6.2(b), a finite set $\mathcal{C}_T$ of correction pairs, and an injective map*

$$\Psi_T : \{0, 1\}^{\mathcal{E}_T} \longrightarrow \{\text{finite legal sets}\}.$$

Put $j_T = |\mathcal{C}_T|$ and $a_T = C_0 + 5m_T + j_T$. Then:

$$(a) \ 0 \leq j_T \leq H(Y_T) \ll T^{1-\varepsilon/2};$$

(b) *for every σ , the set $\Psi_T(\sigma)$ has reciprocal sum r and exactly $a_T + |\sigma|$ components. Every component has length two or three, any two are separated by at least g integers, and*

$$B < \min \Psi_T(\sigma), \quad \max \Psi_T(\sigma) \leq A_1 T^{2-2\varepsilon};$$

(c) *there is an integer τ_T , with $0 \leq \tau_T \leq C_0$ and independent of σ , such that if exactly u coordinates of σ are zero, then $\Psi_T(\sigma)$ has exactly $5u + \tau_T$ components of length three.*

Proof. Take T sufficiently large for Lemma 8.3 and set

$$x_T = r - A_0 - \sum_{d \in \mathcal{E}_T} h(d) = G + 2\eta - \sum_{d \in \mathcal{E}_T} h(d). \quad (8.7)$$

Equation (8.5) gives $2\eta - 1/(KT) < x_T \leq 2\eta$. On increasing the threshold for T so that $1/(KT) < \eta/4$, we obtain

$$\frac{7\eta}{4} < x_T \leq 2\eta. \quad (8.8)$$

The reduced denominator of x_T divides the least common multiple of the denominator of r , the elements of $\mathcal{A}$, and the elements of the unsplit cells Z_d , $d \in \mathcal{E}_T$. The cell elements are Y_T -powersmooth by construction. The other integers are fixed and therefore are Y_T -powersmooth for all sufficiently large T . It follows that $\lambda(x_T) \leq Y_T$. Moreover $x_T > 7\eta/4 > E(L)$ by (6.2).

Apply Theorem 5.1 to x_T , with the fixed L, K and with $Y = Y_T$. It supplies a set $\mathcal{C}_T$ of correction pairs at levels $L < Q \leq Y_T$, using at most $H(Y_T)$ pairs, for which

$$x_f = x_T - \sum_{\mathcal{P} \in \mathcal{C}_T} \rho(\mathcal{P}), \quad x_T - E(L) < x_f \leq x_T, \quad Kx_f \in \mathbb{Z}.$$

By (8.8) and $E(L) < \eta/4$, this remainder satisfies

$$\frac{3\eta}{2} < x_f \leq 2\eta.$$

Choose $\mathcal{D} \subseteq U$ by Lemma 6.2(b) so that $\sum_{d \in \mathcal{D}} 1/(Kd) = x_f$, and put

$$\mathcal{A}'_T = \mathcal{A} \cup \{Kd : d \in \mathcal{D}\}, \quad \tau_T = |\mathcal{D}|.$$

Then $\rho(\mathcal{A}'_T) = A_0 + x_f$, its component count is C_0 , and precisely τ_T of its components are triples. Both $\mathcal{A}'_T$ and $\mathcal{C}_T$ have been chosen without reference to σ .

Define

$$\Psi_T(\sigma) = \mathcal{A}'_T \cup \bigcup_{\mathcal{P} \in \mathcal{C}_T} \mathcal{P} \cup \bigcup_{d \in \mathcal{E}_T} Z_d^{(\sigma_d)}. \quad (8.9)$$

By Proposition 7.3, this set is legal, its components have length two or three, and they have the prescribed separation. Its component count is

$$C_0 + j_T + 5(m_T - |\sigma|) + 6|\sigma| = a_T + |\sigma|,$$

and its reciprocal sum is

$$A_0 + x_f + \sum_{\mathcal{P} \in \mathcal{C}_T} \rho(\mathcal{P}) + \sum_{d \in \mathcal{E}_T} h(d) = r,$$

by the definitions of x_f and x_T . Its triples are precisely the τ_T triples in $\mathcal{A}'_T$ and the five triples of each unsplit cell, proving (c).

The map is injective. For each $d \in \mathcal{E}_T$, the pair $\{6Kd + 48t, 6Kd + 48t + 1\}$ occurs in $\Psi_T(\sigma)$ if and only if $\sigma_d = 1$. Its residue band $[48t, 48t + 1]$ contains no other kind of component, and distinct indices give distinct pairs. The output thus determines every coordinate of σ .

Every reserved element exceeds B . Every correction element exceeds $L^2/5 > B$, by Theorem 5.1(b) and (6.2). Every cell element, in either state, is at least $6KT > B$ for sufficiently large T . For the upper bound, reserved elements are at most the fixed integer $M_0 = K \max U + 2$, cell elements are less than ΓT , and correction elements at level $Q \leq Y_T$ are less than $Q^2 \leq T^{2-2\varepsilon}$, by (4.7). Since $2 - 2\varepsilon > 1$, we can choose A_1 so that

$$\max \Psi_T(\sigma) \leq \max\{M_0, \Gamma T, T^{2-2\varepsilon}\} \leq A_1 T^{2-2\varepsilon}.$$

This completes (b).

Finally, apply Corollary 4.8 with $\nu = \varepsilon/(2(1 - \varepsilon))$. It gives

$$j_T \leq H(Y_T) \ll T^{(1-\varepsilon)(1+\nu)} = T^{1-\varepsilon/2},$$

which is (a). □

Corollary 8.5. *For every sufficiently large integer T and every integer $0 \leq s \leq m_T$, there are at least $\binom{m_T}{s}$ distinct legal sets with reciprocal sum r , exactly $a_T + s$ components, and all the properties in Theorem 8.4(b). In particular every integer in $[a_T, a_T + m_T]$ occurs as a component count.*

Proof. There are $\binom{m_T}{s}$ vectors σ of weight s . Their images under the injective map Ψ_T have the asserted properties by Theorem 8.4. $\square$

9. CONSECUTIVE SCALES AND THE MAIN THEOREMS

The construction at scale T gives an interval of component counts of width $m_T \asymp T$. To apply Proposition 2.1, it remains to show that the selected number of cells changes by $o(T)$ between consecutive scales. The correction count $j_T = o(T)$ already has the required bound; its dependence on T need not be more regular.

Lemma 9.1. *The admissible lists satisfy*

$$|\mathcal{L}_T \Delta \mathcal{L}_{T+1}| = O_{K,C,\varepsilon}(T^\varepsilon).$$

Proof. The index ranges $[T, CT]$ and $[T+1, C(T+1)]$ differ in at most $C+1$ integers. On their intersection, admissibility persists from scale T to scale $T+1$, since $Y_{T+1} > Y_T$. If a cell becomes admissible, one of its fifteen denominators $\alpha d + \beta$ is divisible by a prime power q with

$$Y_T < q \leq Y_{T+1}.$$

But

$$Y_{T+1} - Y_T = (T+1)^{1-\varepsilon} - T^{1-\varepsilon} \leq (1-\varepsilon)T^{-\varepsilon} < 1,$$

so this interval contains at most one integer. If that integer is a prime power q , each fixed linear form has at most

$$\gcd(\alpha, q) \left(\frac{(C-1)T}{q} + 1 \right) \leq 36K((C-1)T^\varepsilon + 1)$$

zeros modulo q in the index range. Summing over the fifteen forms and adding the endpoints proves the bound. $\square$

Lemma 9.2. *For all sufficiently large T ,*

$$|m_{T+1} - m_T| = O_{K,C,\varepsilon}(T^\varepsilon).$$

Proof. Let $\Lambda = \mathcal{L}_T \cap \mathcal{L}_{T+1}$, $P = \mathcal{E}_T$, $P' = \mathcal{E}_{T+1}$ and $s = |\mathcal{L}_T \Delta \mathcal{L}_{T+1}|$. Every relevant index lies in $[T, C(T+1)]$. Thus (7.2) gives

$$h_{\min} := \frac{1}{K(C(T+1)+1)} \leq h(d) \leq \frac{1}{KT} =: h_{\max}, \quad \frac{h_{\max}}{h_{\min}} \leq 3C.$$

Both $P \cap \Lambda$ and $P' \cap \Lambda$ are initial segments of the same ordered set Λ . They are therefore nested. Interchanging P and P' if necessary, suppose that $P \cap \Lambda \subseteq P' \cap \Lambda$, and put $\Delta = (P' \cap \Lambda) \setminus (P \cap \Lambda)$. Then

$$\sum_{d \in P'} h(d) - \sum_{d \in P} h(d) = \sum_{d \in \Delta} h(d) + \sum_{d \in P' \setminus \Lambda} h(d) - \sum_{d \in P \setminus \Lambda} h(d).$$

Each prefix has mass in $[G, G + h_{\max}]$ by (8.5). The left side is consequently less than $h_{\max}$. Since the middle sum on the right is nonnegative and $|P \setminus \Lambda| \leq s$, we obtain

$$|\Delta| h_{\min} \leq \sum_{d \in \Delta} h(d) < (s+1)h_{\max}, \quad |\Delta| \leq 3C(s+1).$$

The parts of the two prefixes outside Λ each contain at most s elements. Hence

$$|m_{T+1} - m_T| \leq |P \Delta P'| \leq 2s + 3C(s+1).$$

Apply Lemma 9.1. $\square$

The comparison is between prefix masses and the number of admissible indices. It requires no estimate for gaps between consecutive admissible indices.

Proof of Theorem B. All parameters in the construction are fixed before T : first g, t, R and r, B, η , then L, K, J , the reserved sets $\mathcal{D}_j$ and their union U , and finally C, ε, Γ . In particular, C_0 and A_0 are fixed. By Theorem 8.4, (8.6) and Lemma 9.2,

$$\kappa(\Psi_T(\sigma)) = 5m_T + b_T + |\sigma|, \quad b_T = C_0 + j_T,$$

where

$$m_T \asymp T, \quad m_{T+1} - m_T = o(T), \quad b_T = o(T).$$

Set $\delta = 2\varepsilon$. The choice (8.3) gives $0 < \delta < 1/5$.

For part (i), apply Proposition 2.1 with $q = 5$. Every sufficiently large k is attained at a scale T with $k \geq 5m_T$, by at least $2^{k/21}$ distinct vectors σ . The sets $\Psi_T(\sigma)$ are distinct by Theorem 8.4; their components have length two or three, exceed the cutoff B , are separated by at least g integers, and have total reciprocal sum r . Their largest denominator is at most $A_1 T^{2-2\varepsilon}$. Since

$$k \geq 5m_T \geq 5KGT,$$

that denominator is at most

$$A_1(5KG)^{-(2-\delta)}k^{2-\delta}.$$

This proves part (i) after enlarging A .

For part (ii), use the upper endpoint of the full count interval,

$$v_T = a_T + m_T = C_0 + 6m_T + j_T.$$

The bounds in Lemma 9.2 and Theorem 8.4 imply

$$|v_T - v_{T-1}| \leq 6|m_T - m_{T-1}| + j_T + j_{T-1} = O(T^{1-\varepsilon/2}).$$

Choose T_* so that the construction and these estimates hold for $T \geq T_*$. For $k > v_{T_*}$, let $T \geq T_*$ be the least index with $v_T \geq k$. Then $T > T_*$ and $v_{T-1} < k \leq v_T$. The nonnegative integer

$$u = v_T - k < v_T - v_{T-1} = O(T^{1-\varepsilon/2})$$

is at most m_T for large k . Choose a vector σ with exactly u zero coordinates. Its set has k components, of which at most

$$5u + C_0 = O(T^{1-\varepsilon/2}) = O(k^{1-\delta/4})$$

are triples, by Theorem 8.4(c) and $k \geq a_T \geq 5m_T \geq 5KGT$. The denominator bound follows as in part (i). A common enlargement of A and k_0 establishes both assertions. $\square$

Proof of Theorem A. Set $r = 1$, $B = 0$ and $g = 1$ in Theorem B(i). $\square$

The freedom to impose a denominator cutoff also permits a prescribed finite configuration to be retained.

Corollary 9.3 (Extension of a finite configuration). *Let $r > 0$ be rational and $g \geq 1$ an integer. Let S_0 be a finite set of positive integers with $\rho(S_0) < r$, whose components have length two or three and are separated by at least g integers. For every sufficiently large integer k there is a set $S \supseteq S_0$ with $\rho(S) = r$ and exactly k components, all of length two or three and separated by at least g integers. All elements of $S \setminus S_0$ may be required to exceed any prescribed bound.*

Proof. Choose B at least the prescribed bound and such that $B \geq n + g$ for every $n \in S_0$. Apply Theorem B with rational $r - \rho(S_0)$, cutoff B , separation g and component count $k - \kappa(S_0)$. Adjoining S_0 preserves every component and gives the required sum and count. This choice of B also covers $S_0 = \emptyset$. $\square$

The thresholds are effective in principle, since the exponential-sum and divisor bounds and the prime estimates in [Lemma 8.2](#) admit effective constants and the construction in [Lemma 6.1](#) terminates; they are large, as the fixed reserve contains $J = \lceil 4K\eta \rceil$ unit-sum groups with $K = \text{lcm}(1, \dots, L)$, the condition $\log C > 2rK$ makes C large and (8.3) then makes ε small. Only the unsplit cells and the completed reserved pairs contribute triples, which gives the sublinear bound in [Theorem B\(ii\)](#); the construction does not give representations by pairs alone for every large count.

APPENDIX A. FINITE REPRESENTATIONS OF 1

This appendix gives exact finite results for representations of 1 by pairwise disjoint, pairwise nonadjacent intervals of length at least two. Intervals longer than three are permitted here. The results follow from exhaustive enumeration using [Lemma 3.2](#) and exact rational arithmetic; the proof of [Theorem B](#) does not depend on them.

Proposition A.1. *(i) No such representation has all denominators at most 84. Exactly four have all denominators at most 85; they have 8, 9, 10 and 11 intervals.
 (ii) Every such representation with all denominators at most 99 has at least eight intervals, and there is one with seven intervals and largest denominator 100.
 (iii) Such representations exist for every number of intervals from 7 to 21.*

Exact verification. A representation cannot contain the denominator 1: its interval would also contain 2, making the reciprocal sum greater than 1. For a denominator bound N , begin with the candidate set $\{2, \dots, N\}$. Repeat two reductions until the set is unchanged. First, remove any candidate with neither neighbour present. Second, for each prime p , let q be the largest power of p dividing a remaining candidate. Compute all subset sums in $\mathbb{F}_p$ of the residues $(n/q)^{-1}$ for candidates n divisible by q , and remove any such n contained in no zero-sum subset. The first reduction follows from the minimum interval length, and the second from [Lemma 3.2](#); neither removes a denominator belonging to an admissible representation.

For each maximal consecutive run of the surviving candidates, enumerate all subsets without isolated elements, including the empty subset. Every legal support arises from exactly one choice in each run, since a selected interval cannot cross a missing candidate. Let D be the least common multiple of the surviving candidates and assign weight D/n to n . A meet-in-the-middle enumeration selects precisely the combined choices with total weight D . Recheck each resulting support in rational arithmetic.

At $N = 84$ the enumeration returns no support. At $N = 85$ it returns exactly the four representations displayed below. At $N = 99$ it returns seven representations, each with at least eight intervals. These counts prove (i) and the first assertion of (ii). The displayed seven-interval identity proves the remaining assertion of (ii). For (iii), the accompanying files give the examples summarized in [Table 1](#); their disjointness, nonadjacency, interval lengths and reciprocal sums are checked exactly. $\square$

The four representations with all denominators at most 85 are

$$\begin{aligned}
 1 &= H[7, 10] + H[14, 15] + H[17, 19] + H[34, 36] + H[44, 45] + H[56, 57] + H[76, 77] + H[84, 85], \\
 1 &= H[5, 6] + H[14, 15] + H[17, 18] + H[20, 22] + H[27, 28] + H[33, 34] + H[44, 45] \\
 &\quad + H[54, 55] + H[84, 85], \\
 1 &= H[6, 8] + H[14, 15] + H[17, 18] + H[26, 27] + H[34, 35] + H[44, 45] + H[54, 56] \\
 &\quad + H[65, 66] + H[77, 78] + H[84, 85], \\
 1 &= H[7, 8] + H[11, 12] + H[17, 18] + H[21, 22] + H[26, 28] + H[33, 34] + H[44, 45] + H[54, 56] \\
 &\quad + H[65, 66] + H[77, 78] + H[84, 85],
 \end{aligned}$$

and a representation with seven intervals is

$$1 = H[4, 5] + H[9, 11] + H[21, 22] + H[25, 26] + H[65, 66] + H[77, 78] + H[99, 100].$$

Consequently, the minimum possible largest denominator is 85, and the minimum for a representation with seven intervals is 100. These computations do not decide whether a representation with fewer than seven intervals exists. The examples with 7, 9, 10, 11, 12 and 20 intervals in Table 1 use only pairs and triples. The other examples contain longer intervals and therefore do not establish the pair-and-triple conclusion of Theorem B at those small values of k . None of the finite examples is needed for its eventual-existence assertion.

k	largest denominator	largest length	k	largest denominator	largest length
7	100	3	15	190	4
8	85	4	16	190	4
9	85	3	17	190	5
10	85	3	18	190	4
11	85	3	19	190	5
12	176	3	20	196	2
13	145	4	21	2080	5
14	196	4			

TABLE 1. Examples for Proposition A.1(iii): number of intervals k , largest denominator, and largest interval length.

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