

ABSTRACT. We prove that if $2 \leq a_1 < a_2 < \dots$ are integers for which $\sum_n 1/a_n$ and $\sum_n 1/(a_n - 1)$ both have rational sums, then $a_n^{1/2^n} \rightarrow 1$. This is the critical-growth assertion of Erdős, left open by the known doubly exponential constructions with both sums rational; no regularity of growth is assumed, and a limit superior greater than one, finite or infinite, is excluded.

We show that a rational reciprocal sum of a nondecreasing sequence $b_j \geq j^2$ with denominator q satisfies $\log b_j \leq 2^{14} 2^j \log(3q)$, uniformly in the sequence and the numerator. Restarting this bound after every finite prefix of the difference series $\sum_n 1/[a_n(a_n - 1)]$ shows that a positive limit superior of $2^{-n} \log a_n$ forces eventual exponential growth; pairing the two tails so that their off-diagonal product becomes a positive integer once the prefix denominators are cleared then gives $\log a_n = O(3^{n/2})$, and $\sqrt{3} < 2$ closes the argument. Doubly exponential growth $a_n^{1/\beta^n} \rightarrow \infty$ with both sums rational is therefore possible only for $\beta < \sqrt{3}$, while it is known for every $\beta < \sqrt{6/5}$. Taken together, the two rational sums bound each other's tails, and the margin between $\sqrt{3}$ and 2 is the whole proof.

1. INTRODUCTION

Erdős asked how rapidly an increasing sequence of integers $1 \leq a_1 < a_2 < \dots$ can grow when the two series

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{a_n - 1} \quad (1.1)$$

both have rational sums [5, p. 64], [3, p. 334], [4, p. 104]; the question is recorded as Problem 265 at <https://www.erdosproblems.com/265>. The second series excludes $a_1 = 1$, so $a_1 \geq 2$, and increasing order is part of the formulation (Remark 7.5). Erdős expected that $a_n^{1/n} \rightarrow \infty$ should be possible, while $a_n^{1/2^n} \rightarrow 1$ should be necessary. The first assertion follows from a theorem of Kovač and Tao [7, §2.2.1 and (7.9)]: for every fixed $d \geq 1$ and every $1 < \beta < ((2d+2)/(2d+1))^{1/d}$ there is a strictly increasing sequence (c_n) with $c_n^{1/\beta^n} \rightarrow \infty$ such that all d sums $\sum_n 1/(c_n + j)$, $0 \leq j < d$, are rational; their construction leaves the critical base 2 unresolved [7, §2.2.1].

Theorem A (Critical growth). *Let $(a_n)_{n \geq 1}$ be a strictly increasing sequence of integers with $a_1 \geq 2$. If both series in (1.1) converge to rational numbers, then*

$$\lim_{n \rightarrow \infty} \frac{\log a_n}{2^n} = 0, \quad \text{equivalently} \quad \lim_{n \rightarrow \infty} a_n^{1/2^n} = 1. \quad (1.2)$$

The theorem permits arbitrarily irregular growth and excludes a limit superior greater than 1, finite or infinite; the equivalent formulation with the sums $\sum_n 1/c_n$ and $\sum_n 1/(c_n + 1)$ is Corollary 7.1. For one reciprocal series, growth at the scale 2^n need not force irrationality: the Sylvester recurrence gives a rational sum and $\lim_n 2^{-n} \log a_n > 0$ (Example 7.4), whereas $\sum_n 1/a_n$ is irrational if $a_n^{1/2^n} \rightarrow \infty$ [6, 2], and Erdős proved that an infinite limit superior suffices under an eventual lower bound $a_n \geq n^{1+\varepsilon}$ [2, Theorem 1]. The proof rests on a denominator-uniform estimate for the rational case.

Theorem B (Uniform denominator bound). *Let $(b_j)_{j \geq 1}$ be a nondecreasing sequence of positive integers with $b_j \geq j^2$ for every j . If*

$$\sum_{j=1}^{\infty} \frac{1}{b_j} = \frac{p}{q}, \quad p, q \in \mathbb{N},$$

then

$$\log b_j \leq C 2^j \log(3q) \quad (j \geq 1), \quad C = 2^{14}. \quad (1.3)$$

The fraction p/q need not be reduced.

The value of C is not optimised; its independence of the sequence and of the numerator is what permits the estimate to be applied after every finite prefix, with the denominator of the remaining tail. The selection argument in its proof follows the peak method of Erdős [2]; a related selection is used by Barreto, Kang, Kim, Kovač and Zhang [1] for the series $\sum_n 1/(a_n a_{n+1})$. For sequences with eventual exponential growth the proof of [Theorem A](#) gives the stronger bound $\log a_n = O(3^{n/2})$, which locates the largest base of doubly exponential growth compatible with two rational sums.

Corollary C (Double-exponential bases). *Let β_* be the supremum of the bases $\beta > 1$ for which some sequence satisfying the hypotheses of [Theorem A](#) has $a_n^{1/\beta^n} \rightarrow \infty$. Then*

$$\sqrt{6/5} \leq \beta_* \leq \sqrt{3},$$

and every admissible base is strictly below $\sqrt{3}$.

If $\sum_j 1/b_j = p/q$, put $t_j = \log b_j$ and $T_m = \sum_{j \leq m} t_j$; clearing the denominator of a positive tail gives $q e^{T_{n-1}} \sum_{j \geq n} 1/b_j \geq 1$, and the lower bound $b_j \geq j^2$ gives $\sum_{j \geq n} 1/b_j \leq 3b_n^{-1/2}$, hence $T_n \leq 3T_{n-1} + 2 \log(3q)$. To obtain (1.3), an index Q is selected at which $t_Q - T_{Q-1}$ is large; up to a later violating index R an exponential envelope bounds the tail from Q , beyond R the polynomial envelope bounds the entire remainder, and the resulting upper bound contradicts the cleared-tail integer. For [Theorem A](#), write $L_n = \log a_n$, $S_m = \sum_{i \leq m} L_i$ and $K = \limsup_n 2^{-n} L_n$. The difference of the two rational sums is $\sum_n 1/[a_n(a_n - 1)]$, with denominators at least n^2 , so [Theorem B](#) gives $K < \infty$; applied after the first m terms are deleted, with the denominator $qr \prod_{i \leq m} a_i(a_i - 1)$ of the difference tail, it gives $S_m \geq (K/C)2^m - \frac{1}{2} \log(3qr)$, so $K > 0$ would force $\log a_m/m \rightarrow \infty$. For the original tails $X_n = \sum_{j \geq n} 1/a_j$ and $Y_n = \sum_{j \geq n} 1/(a_j - 1)$ the difference $Y_n - X_n$ is the diagonal of $X_n Y_n$; the off-diagonal part $E_n = X_n Y_n - Y_n + X_n$ is positive, at most $4X_n X_{n+1}$, and becomes an integer when multiplied by $qr \prod_{i < n} a_i(a_i - 1)$. Under $K > 0$ the exponential tail bounds therefore give $L_n + L_{n+1} \leq 2S_{n-1} + O(n)$, hence $S_{n+1} \leq 3S_{n-1} + O(n)$, and iteration on both parity classes yields $L_n = O(3^{n/2})$; since $\sqrt{3} < 2$, this contradicts $K > 0$.

Sections 2 and 3 prove [Theorem B](#), Sections 4–6 prove [Theorem A](#), Section 7 proves [Corollary C](#) and records equivalent forms and examples, and Appendix A gives exact formulas for the joint integers in Cantor's example.

Cantor observed that $a_n = \binom{n}{2}$, $n \geq 2$, has both sums rational [5, p. 64], and the folklore irrationality of $\sum_n 1/a_n$ under $a_n^{1/2^n} \rightarrow \infty$ bounds the growth from above by doubly exponential; the remaining question was the exact exponent, which [Theorem A](#) settles at 2. For the series $\sum_n 1/(a_n a_{n+1})$ the corresponding critical base is the golden ratio [1].

2. CLEARED TAILS AND TWO ENVELOPES

Lemma 2.1 (The cleared tail). *Let $(b_j)_{j \geq 1}$ be positive integers with $\sum_{j \geq 1} 1/b_j = p/q$, where $p, q \in \mathbb{N}$. Then, for every $n \geq 1$,*

$$q \left(\prod_{i < n} b_i \right) \sum_{j \geq n} \frac{1}{b_j} \in \mathbb{N}. \quad (2.1)$$

In particular, this quantity is at least 1.

Proof. Clearing the denominator after removing the first $n - 1$ terms gives

$$q \left(\prod_{i < n} b_i \right) \sum_{j \geq n} \frac{1}{b_j} = p \prod_{i < n} b_i - q \sum_{k < n} \prod_{\substack{i < n \\ i \neq k}} b_i \in \mathbb{Z}.$$

The remaining tail is positive, so the integer is positive. This also includes $n = 1$, when the expression equals p . $\square$

Lemma 2.2 (Two envelopes). *For real numbers $M \geq 1$ and $t \geq 0$,*

$$\sum_{j=1}^{\infty} \min\{M^{-2}, j^{-2}\} \leq \frac{3}{M}, \quad (2.2)$$

$$\sum_{j=1}^{\infty} \min\{e^{-t}, e^{-j}\} \leq (t+2)e^{-t}. \quad (2.3)$$

Proof. For (2.2), take $h = \lfloor M \rfloor$. Then $M/2 \leq h \leq M$, and the contribution from $j \leq h$ is at most $h/M^2 \leq 1/M$. For the remaining terms,

$$\sum_{j>h} j^{-2} \leq \int_h^{\infty} x^{-2} dx = \frac{1}{h} \leq \frac{2}{M}.$$

For (2.3), take $h = \lfloor t \rfloor$. The contribution from $j \leq h$ is at most te^{-t} . Since $h+1 > t$, the remaining geometric series satisfies

$$\sum_{j \geq h+1} e^{-j} = \frac{e^{-(h+1)}}{1 - e^{-1}} < \frac{e}{e-1} e^{-t} < 2e^{-t}. \quad \square$$

Corollary 2.3 (Tail bounds). *Let $(b_j)_{j \geq 1}$ be a nondecreasing sequence of positive real numbers.*

(i) *If $b_j \geq j^2$ for all j , then*

$$\sum_{j \geq n} \frac{1}{b_j} \leq 3b_n^{-1/2} \quad (n \geq 1). \quad (2.4)$$

It follows that $\sum_{j>R} 1/b_j \leq 3b_R^{-1/2}$ for every $R \geq 1$.

(ii) *If $b_j \geq e^j$ for $Q \leq j \leq R$, where $1 \leq Q \leq R$, then*

$$\sum_{j=Q}^R \frac{1}{b_j} \leq \frac{\log b_Q + 2}{b_Q}. \quad (2.5)$$

If $b_j \geq e^j$ for all $j \geq Q$, the same upper bound holds for $\sum_{j \geq Q} 1/b_j$.

Proof. For $j \geq n$ in part (i), monotonicity gives $b_j \geq \max\{b_n, j^2\}$. Apply (2.2) with $M = \sqrt{b_n}$, extending the majorant to all positive indices. The bound beyond R follows by taking $n = R+1$ and using $b_{R+1} \geq b_R$.

For part (ii), the summand at an index $Q \leq j \leq R$ is at most $\min\{b_Q^{-1}, e^{-j}\}$. Extend this majorant to all $j \geq 1$ and use (2.3) with $t = \log b_Q \geq Q > 0$. The same argument applies to the infinite tail when the exponential lower bound holds for every $j \geq Q$. $\square$

Lemma 2.4 (The preliminary bound). *Under the hypotheses of Theorem B, define*

$$t_j = \log b_j, \quad T_n = \sum_{j=1}^n t_j, \quad T_0 = 0, \quad B = \log(3q).$$

Then $B > 1$, and for every $n \geq 1$,

$$t_n \leq 2T_{n-1} + 2B, \quad (2.6)$$

$$T_n \leq 3T_{n-1} + 2B \leq B(3^n - 1). \quad (2.7)$$

Proof. The cleared-tail integer and (2.4) give

$$1 \leq qe^{T_{n-1}} \sum_{j \geq n} \frac{1}{b_j} \leq 3qe^{T_{n-1}-t_n/2}.$$

Taking logarithms proves (2.6). Adding T_{n-1} gives the recurrence in (2.7); iteration from $T_0 = 0$ then yields

$$T_n \leq 2B \sum_{j=0}^{n-1} 3^j = B(3^n - 1).$$

Finally, $q \geq 1$ implies $B \geq \log 3 > 1$. □

3. THE UNIFORM BOUND AT THE CRITICAL SCALE

Lemma 3.1 (A controlled increase). *Let $R > P \geq 16$ be integers, let $H \geq 1$ and $0 < B \leq H$, and suppose that the nonnegative real numbers $\mu_P, \dots, \mu_R$ satisfy*

$$\mu_P \leq H, \quad \mu_R > 16H, \quad \mu_j \leq \frac{3}{2}\mu_{j-1} + \frac{2B}{2^j} \quad (P < j \leq R).$$

There is an index $P < Q \leq R$ for which

$$\mu_i \leq 2H \quad (P \leq i < Q), \quad (3.1)$$

$$H < \mu_Q < 4H, \quad (3.2)$$

$$\mu_Q - \mu_{Q-1} > \frac{H}{Q^2 + 1}. \quad (3.3)$$

Proof. Choose the first index $P < Q \leq R$, if one exists, satisfying

$$\mu_Q > H, \quad \mu_Q > (1 + Q^{-2}) \max_{P \leq i < Q} \mu_i. \quad (3.4)$$

To establish its existence, put $M_j = \max\{H, \mu_P, \dots, \mu_j\}$. We have $M_P = H$. At each index $j > P$ where the selection condition fails, either $\mu_j \leq H$ or $\mu_j \leq (1 + j^{-2}) \max_{P \leq i < j} \mu_i$. In either case,

$$M_j \leq (1 + j^{-2})M_{j-1}.$$

If no index satisfied (3.4), it would follow that

$$M_R \leq H \prod_{j=P+1}^R (1 + j^{-2}) \leq H \exp\left(\sum_{j>P} j^{-2}\right) \leq He^{1/P} < 2H,$$

contradicting $\mu_R > 16H$. Thus Q exists. Applying the same estimate only up to $Q - 1$ gives $M_{Q-1} < 2H$, and hence (3.1).

Since $Q \geq 17$ and $B \leq H$, the assumed recurrence now gives

$$\mu_Q \leq \frac{3}{2} \cdot 2H + \frac{2H}{2^{17}} < 4H.$$

Together with the first selection condition, this proves (3.2). The second selection condition implies $\mu_{Q-1} < \mu_Q Q^2 / (Q^2 + 1)$, so

$$\mu_Q - \mu_{Q-1} > \frac{\mu_Q}{Q^2 + 1} > \frac{H}{Q^2 + 1}. \quad \square$$

Lemma 3.2 (Numerical bounds). *The following inequalities hold:*

$$(3/2)^{16} < 1024, \quad (3.5)$$

$$P^2 < 2^P \quad (P \geq 16), \quad (3.6)$$

$$2^Q \geq 2Q(Q^2 + 1) \quad (Q \geq 17). \quad (3.7)$$

Proof. For (3.5), compare $3^{16} = 43\,046\,721$ with $1024 \cdot 2^{16} = 67\,108\,864$. For (3.6), check $P = 16$; the quotient $P^2/2^P$ decreases for $P \geq 3$, since $(P+1)^2 < 2P^2$ is equivalent to $(P-1)^2 > 2$. For (3.7), the initial case is $2^{17} = 131\,072 \geq 9860 = 2 \cdot 17(17^2 + 1)$. The quotient $2^Q/[Q(Q^2 + 1)]$ increases for $Q \geq 4$: comparison of consecutive values reduces to

$$2Q(Q^2 + 1) - (Q + 1)((Q + 1)^2 + 1) = Q^3 - 3Q^2 - 2Q - 2 > 0.$$

Indeed, substituting $Q = k + 4$ makes the last polynomial $k^3 + 9k^2 + 22k + 6$, which is positive for $k \geq 0$. $\square$

Proof of Theorem B. Retain the notation of Lemma 2.4, and set

$$H = 1024B, \quad \mu_n = \frac{T_n}{2^n} \quad (n \geq 0).$$

Since $16H = 2^{14} \log(3q)$, it suffices to rule out an index $R \geq 1$ with

$$t_R > 16H2^R. \quad (3.8)$$

Suppose such an index exists. If $R \leq 16$, then (2.7) and (3.5) imply

$$t_R \leq T_R < B3^R = B(3/2)^R 2^R \leq B(3/2)^{16} 2^R < H2^R,$$

a contradiction. Thus $R \geq 17$.

Define

$$P = \max(\{16\} \cup \{j : 16 \leq j < R, t_j < j\}). \quad (3.9)$$

Then $16 \leq P < R$, and

$$t_j \geq j \quad (P < j \leq R). \quad (3.10)$$

For $j < R$, this follows from the choice of P ; for $j = R$, it follows from (3.8). We also have $\mu_P \leq H$. When $P = 16$, this is a consequence of (2.7) and (3.5). When $P > 16$, monotonicity of (t_j) and the inequality $t_P < P$ give $T_P \leq Pt_P < P^2$, so (3.6) gives $\mu_P < P^2/2^P < 1 \leq H$. Dividing the recurrence in (2.7) by 2^j yields

$$\mu_j \leq \frac{3}{2}\mu_{j-1} + \frac{2B}{2^j} \quad (j \geq 1),$$

whereas (3.8) gives $\mu_R > 16H$.

Apply Lemma 3.1 on $[P, R]$. Its index Q satisfies $17 \leq Q \leq R$, and its conclusions become

$$T_{Q-1} \leq H2^Q, \quad (3.11)$$

$$t_Q \leq T_Q < 4H2^Q, \quad (3.12)$$

$$t_Q - T_{Q-1} = T_Q - 2T_{Q-1} = 2^Q(\mu_Q - \mu_{Q-1}) > \frac{H2^Q}{Q^2 + 1}. \quad (3.13)$$

The exponential bound (3.10) holds throughout $Q \leq j \leq R$. Thus (2.5) gives

$$\sum_{j=Q}^R \frac{1}{b_j} \leq (t_Q + 2)e^{-t_Q}. \quad (3.14)$$

For the indices beyond R , the polynomial envelope in Corollary 2.3(i) gives

$$\sum_{j>R} \frac{1}{b_j} \leq 3e^{-t_R/2}. \quad (3.15)$$

Combining these two estimates with Lemma 2.1, we obtain

$$1 \leq qe^{T_{Q-1}} \sum_{j \geq Q} \frac{1}{b_j} \leq q(t_Q + 2)e^{T_{Q-1} - t_Q} + 3qe^{T_{Q-1} - t_R/2}. \quad (3.16)$$

We bound the two terms on the right separately. First, (3.12) and $H \geq 1$ give $t_Q + 2 \leq 7H2^Q$. Since $\log q \leq B \leq H$, $\log H \leq H$, $\log 7 < 2$ and $\log 2 < 1$,

$$\log(q(t_Q + 2)) \leq 2H + 2 + Q. \quad (3.17)$$

On the other hand, (3.13) and (3.7) give $T_{Q-1} - t_Q < -2QH$. Therefore

$$q(t_Q + 2)e^{T_{Q-1} - t_Q} < \exp(2H + 2 + Q - 2QH) \leq \exp(4 - Q) \leq e^{-13} < \frac{1}{4}.$$

Here the middle inequality uses $2H(1 - Q) \leq 2(1 - Q)$, since $H \geq 1$ and $Q > 1$. For the second term, use $3q = e^B$, (3.11), $Q \leq R$ and (3.8) to obtain

$$3qe^{T_{Q-1} - t_R/2} = \exp(B + T_{Q-1} - t_R/2) < \exp(H + H2^R - 8H2^R) \leq e^{-6H2^R} < \frac{1}{4}.$$

The upper bound in (3.16) is consequently less than $1/2$, which is impossible. This rules out (3.8) and proves (1.3). $\square$

4. RESTARTS

The uniformity in [Theorem B](#) permits its application after every finite prefix, with the denominator of the remaining tail included explicitly.

Proposition 4.1 (Restarting after a prefix). *Under the hypotheses of [Theorem B](#), put*

$$t_n = \log b_n, \quad T_m = \sum_{i=1}^m t_i, \quad T_0 = 0, \quad B = \log(3q), \quad \kappa = \limsup_{n \rightarrow \infty} 2^{-n} t_n.$$

Then $0 \leq \kappa < \infty$, and for all integers $m \geq 0$ and $j \geq 1$,

$$t_{m+j} \leq C 2^j (B + T_m). \quad (4.1)$$

Consequently,

$$T_m \geq \frac{\kappa}{C} 2^m - B \quad (m \geq 0). \quad (4.2)$$

If $\kappa > 0$, then $\log b_m/m \rightarrow \infty$.

Proof. [Theorem B](#) gives $t_n \leq CB2^n$, hence $0 \leq \kappa \leq CB$. Fix $m \geq 0$ and set

$$d_j = b_{m+j} \quad (j \geq 1), \quad q_m = q \prod_{i=1}^m b_i.$$

The sequence (d_j) is nondecreasing and satisfies $d_j \geq (m+j)^2 \geq j^2$. Its reciprocal sum is the positive rational tail of the original series, and [Lemma 2.1](#), applied at $n = m+1$, shows that q_m is a denominator of this tail. Applying [Theorem B](#) to (d_j) therefore gives

$$t_{m+j} \leq C 2^j \log(3q_m) = C 2^j (B + T_m),$$

which proves (4.1).

For this fixed m , divide by 2^{m+j} and take the limit superior as $j \rightarrow \infty$. The left side has limit superior κ , so

$$\kappa \leq C 2^{-m} (B + T_m),$$

proving (4.2). For $m \geq 1$, monotonicity gives $T_m \leq m t_m$, and hence

$$\frac{t_m}{m} \geq \frac{\kappa}{C} \frac{2^m}{m^2} - \frac{B}{m^2}.$$

The right side tends to infinity if $\kappa > 0$. $\square$

For the remainder of the proof, assume the hypotheses of [Theorem A](#). Choose positive integers u, v, q, r such that

$$X_1 := \sum_{n \geq 1} \frac{1}{a_n} = \frac{u}{q}, \quad Y_1 := \sum_{n \geq 1} \frac{1}{a_n - 1} = \frac{v}{r}. \quad (4.3)$$

Neither fraction need be reduced. Write

$$L_n = \log a_n, \quad S_m = \sum_{i=1}^m L_i, \quad S_0 = 0, \quad B_0 = \log(3qr). \quad (4.4)$$

Strict increase and $a_1 \geq 2$ imply $a_n \geq n + 1$. Thus

$$b_n := a_n(a_n - 1) \geq n(n + 1) \geq n^2 \quad (4.5)$$

is a strictly increasing sequence of integers, and

$$\sum_{n \geq 1} \frac{1}{b_n} = \sum_{n \geq 1} \left(\frac{1}{a_n - 1} - \frac{1}{a_n} \right) = Y_1 - X_1 = \frac{vq - ur}{qr} > 0.$$

The numerator is a positive integer, so [Theorem B](#) applies with denominator qr . We shall also use

$$\log b_n = 2L_n + \log(1 - 1/a_n), \quad -\log 2 \leq \log(1 - 1/a_n) < 0. \quad (4.6)$$

Proposition 4.2 (Growth from a positive limit superior). *Let $K = \limsup_{n \rightarrow \infty} 2^{-n} L_n$. Then*

$$0 < L_n \leq CB_0 2^n \quad (n \geq 1), \quad (4.7)$$

so $0 \leq K < \infty$, and

$$S_m \geq \frac{K}{C} 2^m - \frac{B_0}{2} \quad (m \geq 0). \quad (4.8)$$

If $K > 0$, then $L_m/m \rightarrow \infty$. In particular, for every $c > 0$ there is an index N_c such that

$$a_m \geq e^{cm} \quad (m \geq N_c). \quad (4.9)$$

Proof. Apply [Proposition 4.1](#) to the sequence (4.5) with denominator qr . Since $b_n \geq a_n$, its uniform bound gives (4.7). By (4.6),

$$\limsup_{n \rightarrow \infty} 2^{-n} \log b_n = 2K.$$

For a prefix of length m , the denominator used in the restart is

$$q_m = qr \prod_{i=1}^m a_i(a_i - 1). \quad (4.10)$$

Writing $T_m = \sum_{i=1}^m \log b_i$, (4.2) gives

$$T_m \geq \frac{2K}{C} 2^m - B_0.$$

Since $T_m \leq 2S_m$ by (4.6), this proves (4.8).

For $m \geq 1$, monotonicity gives $S_m \leq mL_m$, and therefore

$$\frac{L_m}{m} \geq \frac{K}{C} \frac{2^m}{m^2} - \frac{B_0}{2m^2}. \quad (4.11)$$

When $K > 0$, the right side tends to infinity. For any fixed $c > 0$ it is at least c for all sufficiently large m , proving (4.9). $\square$

5. THE JOINT INTEGER OF TWO TAILS

Retain (4.3) and (4.4). For $n \geq 1$, define

$$X_n = \sum_{j \geq n} \frac{1}{a_j}, \quad Y_n = \sum_{j \geq n} \frac{1}{a_j - 1}, \quad (5.1)$$

$$Q_n = q \prod_{j < n} a_j, \quad R_n = r \prod_{j < n} (a_j - 1). \quad (5.2)$$

By Lemma 2.1, $Q_n X_n$ and $R_n Y_n$ are positive integers.

Lemma 5.1 (Diagonal cancellation). *For every $n \geq 1$,*

$$E_n := X_n Y_n - Y_n + X_n = \sum_{\substack{i, j \geq n \\ i \neq j}} \frac{1}{a_i(a_j - 1)} > 0, \quad (5.3)$$

and

$$J_n := Q_n R_n E_n \in \mathbb{N}. \quad (5.4)$$

Proof. The tails converge and have positive terms, so

$$X_n Y_n = \sum_{i, j \geq n} \frac{1}{a_i(a_j - 1)}.$$

The diagonal of this double series is

$$\sum_{j \geq n} \frac{1}{a_j(a_j - 1)} = \sum_{j \geq n} \left(\frac{1}{a_j - 1} - \frac{1}{a_j} \right) = Y_n - X_n.$$

Removing it proves the identity in (5.3). The off-diagonal sum is strictly positive, since it contains positive terms.

Put $A_n = Q_n X_n \in \mathbb{N}$ and $B_n = R_n Y_n \in \mathbb{N}$. Then

$$Q_n R_n E_n = A_n B_n - Q_n B_n + R_n A_n \in \mathbb{Z}.$$

This integer is positive by (5.3), proving (5.4). $\square$

Lemma 5.2 (A bound by consecutive tails). *For every $n \geq 1$,*

$$0 < E_n \leq 4X_n X_{n+1}, \quad (5.5)$$

and consequently

$$1 \leq J_n \leq 4qr e^{2S_{n-1}} X_n X_{n+1}. \quad (5.6)$$

Proof. Set

$$x = \frac{1}{a_n}, \quad y = \frac{1}{a_n - 1}, \quad U = X_{n+1}, \quad V = Y_{n+1}.$$

Thus $X_n = x + U$ and $Y_n = y + V$. Since $y - x = xy$, expansion gives

$$E_n = xV + yU + E_{n+1}. \quad (5.7)$$

Also $E_{n+1} = UV - (V - U) \leq UV$, because $V \geq U$. The inequality $1/(a - 1) \leq 2/a$ for $a \geq 2$ gives $y \leq 2x$ and $V \leq 2U$, hence

$$E_n \leq xV + yU + UV \leq 4xU + 2U^2 \leq 4(x + U)U = 4X_n X_{n+1}.$$

Finally,

$$Q_n R_n = qr \prod_{j < n} a_j(a_j - 1) \leq qr \prod_{j < n} a_j^2 = qr e^{2S_{n-1}}.$$

Combining this with Lemma 5.1 and (5.5) proves (5.6). $\square$

Proposition 5.3 (The recurrence in the exponential regime). *Suppose, in addition to the hypotheses of Theorem A, that there is an integer $N \geq 1$ such that $a_j \geq e^j$ for every $j \geq N$. Put*

$$M = CB_0 + 2, \quad c_0 = \log(4qr) + 2 \log M + \log 2, \quad A = 2 \log 2 + c_0. \quad (5.8)$$

These constants are positive and depend only on q and r . For every $n \geq N$,

$$L_n + L_{n+1} \leq 2S_{n-1} + 2n \log 2 + c_0, \quad (5.9)$$

$$S_{n+1} \leq 3S_{n-1} + 2n \log 2 + c_0. \quad (5.10)$$

In particular,

$$S_k \leq 3S_{k-2} + Ak \quad (k \geq N + 1). \quad (5.11)$$

Proof. For $n \geq N$, the infinite-tail bound in Corollary 2.3(ii), applied to (a_j) , gives

$$X_n \leq (L_n + 2) e^{-L_n}. \quad (5.12)$$

The uniform bound (4.7) also gives $L_n + 2 \leq M2^n$ for every $n \geq 1$. Applying (5.12) at n and $n + 1$ in (5.6), then taking logarithms, yields

$$\begin{aligned} L_n + L_{n+1} &\leq 2S_{n-1} + \log(4qr) + \log(L_n + 2) + \log(L_{n+1} + 2) \\ &\leq 2S_{n-1} + \log(4qr) + 2 \log M + (2n + 1) \log 2. \end{aligned}$$

This is (5.9). Adding S_{n-1} gives (5.10). With $k = n + 1$, the inequality $2(k - 1) \log 2 + c_0 \leq Ak$ proves (5.11). $\square$

6. THE PARITY ITERATION AND THE PROOF OF THEOREM A

Lemma 6.1 (The two parity classes). *Let $N \geq 1$ and $A > 0$, and let $(S_n)_{n \geq 0}$ be nonnegative real numbers satisfying $S_k \leq 3S_{k-2} + Ak$ for every $k \geq N + 1$. For $m \in \{N, N + 1\}$ and every $j \geq 0$,*

$$S_{m+2j} \leq 3^j \left(S_m + A \left(\frac{m}{2} + \frac{3}{2} \right) \right). \quad (6.1)$$

Thus, with

$$D = \max_{m \in \{N, N+1\}} 3^{-m/2} \left(S_m + A \left(\frac{m}{2} + \frac{3}{2} \right) \right), \quad (6.2)$$

one has $S_n \leq D3^{n/2}$ for every $n \geq N$.

Proof. Iteration at $m + 2, m + 4, \dots, m + 2j$ gives

$$\frac{S_{m+2j}}{3^j} \leq S_m + A \sum_{\ell=1}^j \frac{m+2\ell}{3^\ell} \leq S_m + A \left(\frac{m}{2} + \frac{3}{2} \right),$$

where we used $\sum_{\ell \geq 1} 3^{-\ell} = 1/2$ and $\sum_{\ell \geq 1} \ell 3^{-\ell} = 3/4$. This proves (6.1), including $j = 0$. Every $n \geq N$ has a unique expression $n = m + 2j$ with $m \in \{N, N + 1\}$ and $j \geq 0$. Since $3^j = 3^{n/2} 3^{-m/2}$, the definition of D gives the result. $\square$

Corollary 6.2 (Sequences with eventual exponential growth). *Under the hypotheses and notation of Proposition 5.3,*

$$0 < L_n \leq S_n \leq D3^{n/2} \quad (n \geq N),$$

where D is given by (6.2), with A as in (5.8). In particular, $\lim_{n \rightarrow \infty} 2^{-n} \log a_n = 0$.

Proof. Apply Lemma 6.1 to (5.11). Since $L_i > 0$ for every i , we have $L_n \leq S_n$, and therefore

$$0 \leq 2^{-n} L_n \leq D(\sqrt{3}/2)^n \rightarrow 0. \quad \square$$

Proof of Theorem A. Choose q, r as in (4.3) and put $K = \limsup_{n \rightarrow \infty} 2^{-n} L_n$. By Proposition 4.2, $0 \leq K < \infty$. Suppose that $K > 0$. Taking $c = 1$ in Proposition 4.2, we obtain an index N such that $a_j \geq e^j$ for every $j \geq N$. Hence Corollary 6.2 applies and gives $2^{-n} L_n \rightarrow 0$, a contradiction. Thus $K = 0$. Since $2^{-n} L_n \geq 0$, its limit superior being zero implies

$$\lim_{n \rightarrow \infty} \frac{\log a_n}{2^n} = 0.$$

Exponentiation gives (1.2). □

7. EQUIVALENT FORMS AND GROWTH EXAMPLES

Corollary 7.1 (Positive shift). *Let $1 \leq c_1 < c_2 < \dots$ be integers. If both $\sum_n 1/c_n$ and $\sum_n 1/(c_n + 1)$ have rational sums, then $c_n^{1/2^n} \rightarrow 1$.*

Proof. Apply Theorem A to $a_n = c_n + 1$. Since $0 < \log(c_n + 1) - \log c_n \leq \log 2$, the difference of the normalised logarithms tends to zero. □

Corollary 7.2 (Irrationality alternative). *Let $2 \leq a_1 < a_2 < \dots$ be integers with $\sum_n 1/a_n < \infty$. If $\limsup_{n \rightarrow \infty} a_n^{1/2^n} > 1$, then at least one of $\sum_n 1/a_n$ and $\sum_n 1/(a_n - 1)$ is irrational.*

Proof. The second series converges because $1/(a_n - 1) \leq 2/a_n$. Rationality of both sums would contradict Theorem A. □

Example 7.3 (Cantor's sequence). The example of David Cantor recorded by Erdős [5, p. 64], [7, §2.2.1] is, in our indexing,

$$a_n = \binom{n+2}{2} = \frac{(n+1)(n+2)}{2}, \quad n \geq 1.$$

Since $a_n - 1 = n(n+3)/2$,

$$\frac{1}{a_n} = 2 \left(\frac{1}{n+1} - \frac{1}{n+2} \right), \quad \frac{1}{a_n - 1} = \frac{2}{3} \left(\frac{1}{n} - \frac{1}{n+3} \right).$$

The tails therefore have the closed forms

$$X_n = \frac{2}{n+1}, \quad Y_n = \frac{2}{3} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} \right), \quad (7.1)$$

and $X_1 = 1$, $Y_1 = 11/9$. Subtracting and multiplying these expressions, with $t = n+1$, gives

$$Y_n - X_n = \frac{4}{3n(n+1)(n+2)}, \quad E_n = \frac{4(3t^2 - t - 1)}{3t^2(t^2 - 1)}. \quad (7.2)$$

The numerator in the latter expression is positive for $t \geq 2$. Here $\log a_n = O(\log n)$, so the rationality hypotheses do not imply eventual exponential growth.

Example 7.4 (A single rational sum). Let $s_1 = 3$ and $s_{n+1} = s_n^2 - s_n + 1$. The first terms are 3, 7, 43, 1807. The identity $s_{n+1} - 1 = s_n(s_n - 1)$ gives

$$\frac{1}{s_n} = \frac{1}{s_n - 1} - \frac{1}{s_{n+1} - 1}.$$

The sequence increases to infinity, so telescoping yields $\sum_{n \geq 1} 1/s_n = 1/2$. For $s \geq 3$,

$$\frac{2}{3}s^2 \leq s^2 - s + 1 < s^2.$$

Thus $u_n = 2^{-n} \log s_n$ decreases, while

$$u_{n+1} \geq u_n + 2^{-(n+1)} \log(2/3).$$

Consequently,

$$u_n \geq \frac{\log 3}{2} + \log(2/3) \sum_{j=1}^{n-1} 2^{-(j+1)} \geq \frac{\log 2}{2}.$$

It follows that u_n has a positive limit and $\lim_n s_n^{1/2^n} \geq \sqrt{2}$. The second rationality hypothesis in [Theorem A](#) is therefore essential. By [Corollary 7.2](#),

$$\sum_{n \geq 1} \frac{1}{s_n - 1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{42} + \frac{1}{1806} + \cdots$$

is irrational.

Remark 7.5 (Increasing order). The restriction to increasing sequences cannot be replaced by pairwise distinctness. To see this, let (a_j) be Cantor's sequence and construct a permutation π of $\mathbb{N}$ as follows. At position $2k-1$, take the least unused integer. At position $2k$, take any unused integer at least $2^{k2^{2k}}$. After the first k odd choices, every integer at most k has been used, so this construction is indeed a permutation. The reordered sequence $d_n = a_{\pi(n)}$ has the same two rational sums, since both series have positive terms, but

$$d_{2k}^{1/2^{2k}} \geq \pi(2k)^{1/2^{2k}} \geq 2^k \rightarrow \infty.$$

Proof of Corollary C. The lower bound is the case $d = 2$ of the construction of Kovač and Tao [7, §2.2.1 and (7.9)], which gives, for every $1 < \beta < \sqrt{6/5}$, a strictly increasing sequence (c_n) with $c_n^{1/\beta^n} \rightarrow \infty$ and $\sum_n 1/c_n, \sum_n 1/(c_n + 1)$ rational; $a_n = c_n + 1$ then satisfies the hypotheses of [Theorem A](#), and $\log(c_n + 1) - \log c_n \leq \log 2$ does not affect the limit. For the upper bound suppose that $\beta > 1$ and $a_n^{1/\beta^n} \rightarrow \infty$, that is $L_n/\beta^n \rightarrow \infty$. Since $\beta^n/n \rightarrow \infty$, we have $L_n \geq n$ for all sufficiently large n , so [Corollary 6.2](#) gives a constant D with $L_n \leq D(\sqrt{3})^n$ eventually. If $\beta \geq \sqrt{3}$, then $L_n/\beta^n \leq D(\sqrt{3}/\beta)^n \leq D$, contradicting the assumed divergence. Hence every admissible base satisfies $\beta < \sqrt{3}$, and $\beta_* \leq \sqrt{3}$. $\square$

Remark 7.6 (The subcritical exponent). The unconditional conclusion of [Theorem A](#) is $\log a_n = o(2^n)$. The stronger estimate $\log a_n = O(3^{n/2})$ in [Corollary 6.2](#) assumes eventual exponential growth; in the proof of the main theorem this assumption follows from $K > 0$, and it is not a conclusion asserted for every sequence with the two rational sums. The supremum β_* of [Corollary C](#) may equal $\sqrt{3}$, although every admissible base lies strictly below it; we do not determine β_* . For the different series $\sum_n 1/(a_n a_{n+1})$ the corresponding critical base is the golden ratio φ : the sum is irrational when $\limsup_n a_n^{1/\varphi^n} = \infty$, whereas every prescribed finite limit $\lim_n a_n^{1/\varphi^n} > 1$ is compatible with a rational sum [1].

APPENDIX A. JOINT INTEGERS FOR CANTOR'S SEQUENCE

For Cantor's sequence the integer in (5.4) can be written explicitly. These formulas also give examples of every restart and of the consecutive-tail bound.

Proposition A.1. *Let $a_n = \binom{n+2}{2}$ and take $q = 1, r = 9$. For every $n \geq 1$,*

$$Q_n = \frac{n!(n+1)!}{2^n}, \quad R_n = \frac{3(n-1)!(n+2)!}{2^n},$$

and

$$J_n = \frac{((n-1)!n!)^2(3n^2 + 5n + 1)}{4^{n-1}}.$$

Proof. The definitions of Q_n and R_n give

$$Q_n = \prod_{j < n} \frac{(j+1)(j+2)}{2} = \frac{n!(n+1)!}{2^n}, \quad R_n = 9 \prod_{j < n} \frac{j(j+3)}{2} = \frac{3(n-1)!(n+2)!}{2^n}.$$

By (7.2), with $t = n + 1$,

$$E_n = \frac{4(3t^2 - t - 1)}{3t^2(t^2 - 1)} = \frac{4(3n^2 + 5n + 1)}{3(n+1)^2 n(n+2)}.$$

Multiplying the three expressions and cancelling the factors $n(n+1)^2(n+2)$ gives the formula for J_n . $\square$

n	Q_n	R_n	E_n	J_n
1	1	9	1	9
2	3	18	23/54	23
3	18	90	43/180	387
4	180	810	23/150	22 356
5	2 700	11 340	101/945	3 272 400
6	56 700	226 800	139/1764	1 013 310 000

TABLE 1. The first joint integers of Cantor's sequence, with $q = 1$ and $r = 9$. In every row, $J_n = Q_n R_n E_n \in \mathbb{N}$ and $J_n \leq 4Q_n R_n X_n X_{n+1}$.

The accompanying programs verify these formulas, the finite-tail identities, and the elementary numerical inequalities used in the proof. Their inputs, arithmetic conventions and recorded outputs are provided with the source. The universal estimates in [Theorems A](#) and [B](#) are established by the arguments above and do not depend on a finite computation.

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