

SEVENTEEN SQUARES IN THE UNIT SQUARE: ERDŐS PROBLEM #106

PINGYOU LTD

ABSTRACT. We prove that seventeen squares with pairwise disjoint interiors can be drawn in the unit square with total side length exceeding four. This answers in the negative the question of Erdős whether $f(k^2 + 1) = k$, where $f(n)$ is the largest total side length of n such squares, and by Praton's reduction it refutes the conjecture of Erdős–Soifer and Campbell–Staton for every value of its parameter. Sixteen squares are aligned with the unit square and one is rotated by a rational angle, and two supporting lines prove that it overlaps none of its neighbours, so the packing is checked by hand.

We show that the two contacts behind the rotation determine one packing for every angle up to a quarter turn, with total side length an explicit rational function of the half-angle tangent. It exceeds four exactly for angles below 11.45° , and its maximum $4 + \varepsilon^*$, $\varepsilon^* = 1.2432\dots \times 10^{-4}$, occurs at $\theta^* = 7.106^\circ$, where $\tan(\theta^*/2)$ has degree nine with Galois group S_9 , so neither $\cos \theta^*$ nor the optimal length is expressible in radicals. For every $k \geq 4$, $f(k^2 + 1) \geq k + 4\varepsilon^*/k$. Taken together, one tilted square in a hole of the grid defeats the conjecture, and the tilt that gains most is determined exactly.

1. INTRODUCTION

Draw n squares inside the unit square with no common interior point, and let $f(n)$ be the supremum of the sum of their side lengths. Erdős asked whether $f(k^2 + 1) = k$ for every positive integer k , a conjecture he made in the 1930s [5, p. 120], [4]. The question is recorded as Problem 106 at <https://www.erdosproblems.com/106>. The squares are closed, they may touch, and their sides need not be parallel to those of the unit square. Since the areas of the squares sum to at most one, the Cauchy–Schwarz inequality gives $f(k^2) = k$, and the grid of k^2 squares of side $1/k$ attains it. Replacing one square of the grid by two squares of side $1/(2k)$ gives $f(k^2 + 1) \geq k$, and the conjecture asserts that nothing better is possible. It holds for $k = 1$, which is the statement $f(2) = 1$ of Erdős, and for $k = 2$, which Newman proved [5, p. 120]. Halász [7] gave constructions for every number of squares between consecutive squares, and Erdős and Soifer [6] and Campbell and Staton [1] proved

$$f(k^2 + 2c + 1) \geq k + \frac{c}{k} \quad (-k < c < k) \quad (1.1)$$

and conjectured that equality holds. Praton [8] proved that equality in (1.1) for all $k \geq |c|$ at one value of c implies equality at every value of c , so that the whole conjecture is equivalent to the case $c = 0$. For squares whose sides are parallel to those of the unit square, Staton and Tyler [9] introduced the analogous function g , and Baek, Koizumi and Ueoro (arXiv:2411.07274, 2024) proved $g(k^2 + 2c + 1) = k + c/k$ for all $-k < c < k$. The conjecture of Erdős is therefore a statement about rotated squares, and one rotated square refutes it.

Write $\mathcal{U}(x, y; \ell) = [x, x + \ell] \times [y, y + \ell]$ for the aligned square of side ℓ with lower-left corner (x, y) . Define the rational parameters

$$c = \frac{143}{145}, \quad s = \frac{24}{145}, \quad (a, h, b, r) = \frac{(1016863, 449731, 567132, 390485)}{2306608}, \quad (1.2)$$

which satisfy $c^2 + s^2 = 1$, $b = a - h$ and $r(c + s) = h$, and define $d = (1 - a)/2$.

Theorem A (Seventeen squares). The sixteen aligned squares of Table 1 and the rotated square

$$R = (1 - 3h + rs, 1 - a) + \{u(c, s) + v(-s, c) : 0 \leq u, v \leq r\} \quad (1.3)$$

lie in the unit square, have pairwise disjoint interiors, and have total side length

$$4 + \frac{101}{1153304}.$$

In particular $f(17) > 4$, and the conjecture $f(k^2 + 1) = k$ fails at $k = 4$.

Label	Indices	Square	Count
L_j	$0 \leq j \leq 3$	$\mathcal{U}(0, j/4; 1/4)$	4
B_j	$0 \leq j \leq 2$	$\mathcal{U}(1/4, j(a - 1/4); a - 1/4)$	3
C_{ij}	$i, j \in \{0, 1\}$	$\mathcal{U}(a + id, jd; d)$	4
D	—	$\mathcal{U}(1/4, 1/4 + 2b; 3/4 - 2b)$	1
E_i	$i \in \{0, 1\}$	$\mathcal{U}(1 - 2b + ib, 1 - b; b)$	2
F_i	$i \in \{0, 1\}$	$\mathcal{U}(1 - 2h + ih, 1 - a; h)$	2

TABLE 1. The sixteen aligned squares of the layout, in terms of the parameters a , h , $b = a - h$ and $d = (1 - a)/2$. Together with the rotated square R of (1.3) they form the packing of Figure 1.

The rotated square R is the square of side r whose lowest vertex is $(1 - 3h + rs, 1 - a)$ and whose sides make the angle θ with the axes, where $\cos \theta = c$ and $\sin \theta = s$, so $\tan(\theta/2) = 1/12$ and $\theta = 9.527\dots^\circ$. It occupies the hole left by the aligned squares between the column B , the square D , the block C , the row E and the row F (Figure 1). The corner $P = (a, 3a - 3/4)$ of B_2 and the corner $Q = (1 - 2b, 1/4 + 2b)$ of D protrude into the bounding box of R , and the parameters are chosen so that P lies on the line through the lower-left edge of R and Q on the line through its upper-left edge. These two lines separate R from the column B and from D , and every other pair of squares is separated by a horizontal or vertical line. The proof of Theorem A consists of eight rational inequalities and two rational identities.

The two contacts determine the layout for every angle. Let $\theta \in [0, \pi/2]$, $c = \cos \theta$, $s = \sin \theta$ and $t = \tan(\theta/2)$, so that $0 \leq t \leq 1$. For the functionals $\alpha(x, y) = cx + sy$ and $\beta(x, y) = -sx + cy$ the contact conditions read

$$\alpha(P) = \min_R \alpha, \quad \beta(Q) = \max_R \beta, \quad r(c + s) = h, \quad (1.4)$$

three equations that are linear in (a, h, r) once θ is fixed. Define

$$D(t) = 6 + 31t + 56t^2 + 18t^3 - 56t^4 + 31t^5 - 6t^6. \quad (1.5)$$

Theorem B (The contact family). For every $\theta \in [0, \pi/2]$ the system (1.4) has the unique solution

$$a = \frac{(c + s)(21c^2 + 34cs + 35s^2)}{4\Delta}, \quad h = \frac{(c + s)(9c^2 + 17cs + 14s^2)}{4\Delta}, \quad r = \frac{9c^2 + 17cs + 14s^2}{4\Delta}, \quad (1.6)$$

where $\Delta = 12c^3 + 31c^2s + 37cs^2 + 20s^3$. With these parameters the seventeen squares of Table 1 and (1.3) lie in the unit square with pairwise disjoint interiors, and their total side length is $4 + E(t)$ with

$$E(t) = \frac{t^2(3 - 34t + 44t^2 - 34t^3 + 9t^4)}{4D(t)}. \quad (1.7)$$

The excess $E(t)$ is positive exactly for $0 < t < t_1$, where $t_1 = 0.100263\dots$ is the root of $3 - 34t + 44t^2 - 34t^3 + 9t^4$ in $(0, 1)$. On $[0, 1]$ it attains its maximum

$$\varepsilon^* = 0.000124324784811771\dots$$

only at $t^* = 0.0620925\dots$, the unique root in $(0, 1)$ of the irreducible polynomial

$$P(t) = 36 - 519t - 1052t^2 + 1114t^3 + 1372t^4 - 5708t^5 + 2972t^6 + 414t^7 - 480t^8 + 75t^9. \quad (1.8)$$

The numbers t^* , $\cos \theta^*$ and ε^* generate the same field of degree nine over $\mathbb{Q}$, the Galois group of its normal closure is S_9 , and none of them lies in a radical extension of $\mathbb{Q}$. In particular $f(17) \geq 4 + \varepsilon^*$.

The member at $t = 0$ is an aligned configuration with total side length exactly four, which is the maximum for aligned squares by the theorem of Baek, Koizumi and Ueoro, and $E(t) = t^2/8 + O(t^3)$, so every small rotation gains. The angle t^* corresponds to $\theta^* = 7.10615\dots^\circ$, and the rational member of [Theorem A](#) is the member at $t = 1/12$. For fixed θ the total side length is a linear function of (a, h) that increases in both variables, and the two contacts are the constraints that stop it ([Proposition 4.8](#)).

Theorem C (Every $k \geq 4$). For every integer $k \geq 4$,

$$f(k^2 + 1) \geq k + \frac{4\varepsilon^*}{k} \geq k + \frac{101}{288326k} > k.$$

Corollary D (The general conjecture). For every integer c there is an integer $k \geq |c|$ with $f(k^2 + 2c + 1) > k + c/k$.

[Theorem C](#) rests on the grid extension that Praton [8] used in the opposite direction: a $k \times k$ grid whose lower-left 4×4 block is replaced by the packing of [Theorem B](#), scaled by $4/k$, has $k^2 + 1$ squares and total side length $k + 4\varepsilon^*/k$. The same extension shows that $k(f(k^2 + 1) - k)$ is nondecreasing in k , an observation of Singh (arXiv:2601.22163, 2026), and the cases $k \leq 3$ are untouched by it. [Corollary D](#) is Praton's reduction applied to [Theorem A](#).

Section 2 defines the layout and proves the separation lemmas, Section 3 proves [Theorem A](#), Section 4 determines the contact family and proves [Theorem B](#), Section 5 proves [Theorem C](#) and [Corollary D](#), and Appendix A records the exact computations behind the root isolations and the Galois group.

The first packing with $f(17) > 4$ was posted by Silverstein on 29 July 2026 (<https://github.com/Sprite143/erdos-106-counterexample>, AI-generated, with an exact separating-axis verification of all 136 pairs of squares), and the problem's page records the negative answer. That packing is the member of the family of [Theorem B](#) at $t = 1/16$, with $\cos \theta = 255/257$ and total side length $4 + 68073/547596200$, which lies within 1.3×10^{-8} of the maximum $4 + \varepsilon^*$. The member at $t = 1/12$ in [Theorem A](#) has the smallest denominators among the members of [Table 2](#), and its separation is proved by two lines instead of a computation over the pairs. [Theorem B](#) places both packings in one family and determines its optimum. Seamons (<https://doi.org/10.5281/zenodo.21938978>, August 2026) reports twenty-six squares with total side length $5 + 23/894225$, which is a smaller excess at $k = 5$ than [Theorem C](#) gives, and an example of Bašić for $k = 16$, reported by Cambie on the problem's page on 12 August 2026, places a square rotated by 45° among 256 aligned squares to give $f(257) \geq 16 + (10\sqrt{2} - 14)/528$, which we have verified and which is a larger excess at $k = 16$ than [Theorem C](#) gives.

2. THE LAYOUT AND ITS SEPARATIONS

Let a and h be real numbers, define $b = a - h$ and $d = (1 - a)/2$, and let $\theta \in [0, \pi/2]$ with $c = \cos \theta$ and $s = \sin \theta$, so that $c, s \geq 0$ and $c + s \geq 1$. Define $r = h/(c + s)$, $x_0 = 1 - 3h$ and $y_0 = 1 - a$. The layout $\mathcal{L}(a, h, \theta)$ consists of the sixteen aligned squares of [Table 1](#) and the rotated square

$$R = V_0 + \{u(c, s) + v(-s, c) : 0 \leq u, v \leq r\}, \quad V_0 = (x_0 + rs, y_0). \quad (2.1)$$

Since (c, s) and $(-s, c)$ are orthonormal, R is a square of side r whenever $r > 0$, with vertices

$$V_0 = (x_0 + rs, y_0), \quad V_1 = (x_0 + h, y_0 + rs), \quad V_2 = (x_0 + rc, y_0 + h), \quad V_3 = (x_0, y_0 + rc), \quad (2.2)$$

where $rc + rs = h$ was used. Since $c, s \geq 0$, the bounding box of R is

$$[x_0, x_0 + h] \times [y_0, y_0 + h] = [1 - 3h, 1 - 2h] \times [1 - a, 1 - b]. \quad (2.3)$$

Definition 2.1. The layout $\mathcal{L}(a, h, \theta)$ is admissible if the eight inequalities

$$\frac{1}{4} < a, \quad 0 < h \leq \frac{1}{4}, \quad h < b, \quad b < \frac{3}{8}, \quad a + 2h \leq 1, \quad \frac{3}{4} \leq a + 2b \leq 1 \quad (2.4)$$

hold, together with the two contact inequalities

$$\alpha(P) \leq \alpha(V_0), \quad \beta(Q) \geq \beta(V_2), \quad (2.5)$$

where $\alpha(x, y) = cx + sy$, $\beta(x, y) = -sx + cy$, $P = (a, 3a - 3/4)$ and $Q = (1 - 2b, 1/4 + 2b)$.

In terms of the parameters, $\alpha(V_0) = cx_0 + sy_0 + crs$ and $\beta(V_2) = -sx_0 + cy_0 + rc^2$, since $V_2 = V_0 + r(c, s) + r(-s, c)$ gives $\beta(V_2) = \beta(V_0) + r = -sx_0 - rs^2 + cy_0 + r$. The point P is the upper-right corner of B_2 and Q is the lower-right corner of D .

Lemma 2.2 (Containment). *Assume (2.4). Then every square of $\mathcal{L}(a, h, \theta)$ has positive side and lies in $[0, 1]^2$.*

Proof. The sides are $1/4$, $a - 1/4$, d , $3/4 - 2b$, b , h and r . They are positive by (2.4), because $a \leq 1 - 2h < 1$ gives $d > 0$, and $b > h > 0$ gives $b > 0$ and $r = h/(c + s) > 0$. The column L lies in $[0, 1/4] \times [0, 1]$. The column B lies in $[1/4, a] \times [0, 3a - 3/4]$, and $3a \leq 1 + 2h \leq 3/2$ by $a + 2b \leq 1$ and $h \leq 1/4$, so $3a - 3/4 \leq 3/4$. The block C fills $[a, 1] \times [0, 1 - a]$. The square D lies in $[1/4, 1 - 2b] \times [1/4 + 2b, 1]$, where $1 - 2b > 1/4$ by $b < 3/8$. The row E lies in $[1 - 2b, 1] \times [1 - b, 1]$ and the row F in $[1 - 2h, 1] \times [1 - a, 1 - b]$. The square R lies in its bounding box (2.3), which lies in $[0, 1]^2$ because $3h \leq 3/4$ and $0 < b < a < 1$. $\square$

Lemma 2.3 (Aligned separations). *Assume (2.4). Any two of the sixteen aligned squares have disjoint interiors, and the bounding box (2.3) of R has disjoint interior from every aligned square other than B_0 , B_1 , B_2 and D .*

Proof. Two closed squares on opposite sides of a line have disjoint interiors, and the boxes listed in the proof of Lemma 2.2 supply such a line for every pair. Squares of the same group are stacked in a column, a row or a grid. The column L lies in $x \leq 1/4$, and every other square lies in $x \geq 1/4$, because $a \geq 1/4$, $1 - 2b \geq 1/4$, $1 - 2h \geq 1/4$ and $1 - 3h \geq 1/4$. Among the remaining groups, B lies left of C by the line $x = a$, left of E by $x = a \leq 1 - 2b$, which is $a + 2b \leq 1$, and left of F by $x = a \leq 1 - 2h$ and below D by $y = 3a - 3/4 \leq 1/4 + 2b$, which are both $a + 2h \leq 1$. The block C lies below D by $y = 1 - a \leq 1/4 + 2b$, which is $a + 2b \geq 3/4$, below E by $y = 1 - a \leq 1 - b$, and below F by $y = 1 - a$. The square D lies left of E by $x = 1 - 2b$ and left of F by $x = 1 - 2b \leq 1 - 2h$, which is $h \leq b$. The row F lies below E by $y = 1 - b$. These are the ten pairs of distinct groups. The bounding box of R lies in $x \geq 1 - 3h \geq 1/4$, so it is right of L , and the lines $y = 1 - a$, $y = 1 - b$ and $x = 1 - 2h$ separate it from C , E and F . $\square$

Lemma 2.4 (Supporting lines). *Assume (2.4). Then*

$$\max_{B_0 \cup B_1 \cup B_2} \alpha = \alpha(P), \quad \min_R \alpha = \alpha(V_0), \quad \min_D \beta = \beta(Q), \quad \max_R \beta = \beta(V_2).$$

If $\alpha(P) \leq \alpha(V_0)$, then R and each B_j have disjoint interiors, and if $\beta(Q) \geq \beta(V_2)$, then R and D have disjoint interiors.

Proof. On the box $[1/4, a] \times [0, 3a - 3/4]$ that contains the column B , the functional α is nondecreasing in both coordinates because $c, s \geq 0$, so its maximum is at the upper-right corner P . On R , the parametrisation (2.1) gives $\alpha(V_0 + u(c, s) + v(-s, c)) = \alpha(V_0) + u$, with minimum $\alpha(V_0)$ at $u = 0$. The functional β is nonincreasing in x and nondecreasing in y , so on $D = [1/4, 1 - 2b] \times [1/4 + 2b, 1]$ its minimum is at the lower-right corner Q , and on R it equals $\beta(V_0) + v$, with maximum $\beta(V_0) + r = \beta(V_2)$ at $v = r$.

An interior point (x, y) of B_j has $x < a$ and $y < 3a - 3/4$, so $\alpha(x, y) < \alpha(P)$ because $(c, s) \neq (0, 0)$ has nonnegative coordinates. An interior point of R has $u > 0$, so its value of α exceeds $\alpha(V_0)$. When $\alpha(P) \leq \alpha(V_0)$, the interiors lie in the opposite open half-planes $\{\alpha < \alpha(V_0)\}$ and $\{\alpha > \alpha(V_0)\}$. In

the same way, an interior point of D has $x < 1 - 2b$ and $y > 1/4 + 2b$, so its value of β exceeds $\beta(Q)$, while an interior point of R has $v < r$ and its value of β is less than $\beta(V_2)$. When $\beta(Q) \geq \beta(V_2)$ the two interiors lie in opposite open half-planes of the line $\beta = \beta(V_2)$. $\square$

Lemma 2.5 (Total side length). *The sum of the seventeen side lengths of $\mathcal{L}(a, h, \theta)$ is $3 + a + 2h + r$.*

Proof. The sum is $4 \cdot \frac{1}{4} + 3(a - \frac{1}{4}) + 4d + (\frac{3}{4} - 2b) + 2b + 2h + r$, and $4d = 2 - 2a$. $\square$

Proposition 2.6 (Admissible layouts are packings). *If $\mathcal{L}(a, h, \theta)$ is admissible, then its seventeen squares lie in the unit square, have pairwise disjoint interiors, and have total side length $3 + a + 2h + r$.*

Proof. Lemma 2.2 gives the containment, Lemma 2.3 separates every pair of squares except the pairs (R, B_j) and (R, D) , and Lemma 2.4 separates those under the contact inequalities (2.5). The total is Lemma 2.5. $\square$

3. A RATIONAL PACKING OF SEVENTEEN SQUARES

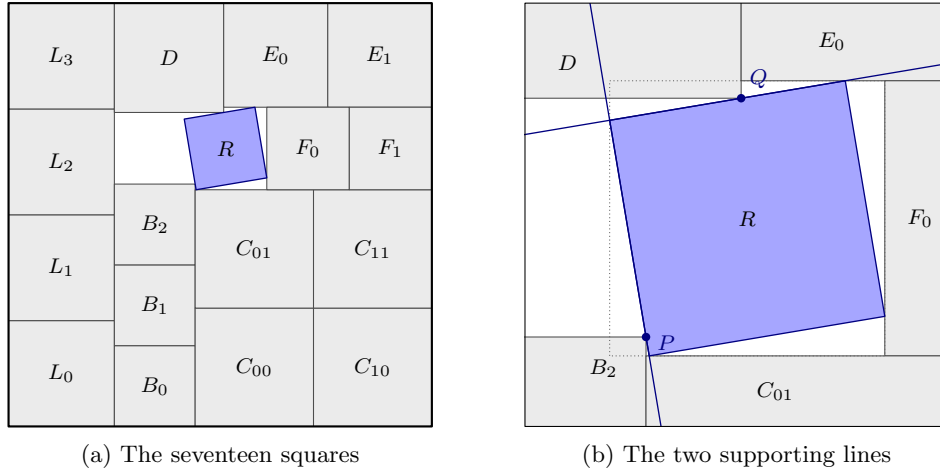


FIGURE 1. The packing of Theorem A and its two supporting lines. The aligned squares are light gray and R is blue. The right panel enlarges the hole: P is the upper-right corner of B_2 , Q is the lower-right corner of D , the dotted square is the bounding box (2.3), and the two lines through the left edges of R separate it from the column B and from D .

Proof of Theorem A. Let a, h, b, r, c and s be the rational numbers in (1.2), and write $N = 2306608$ for their common denominator. Then $b = a - h$ because $1016863 - 449731 = 567132$, and $r(c + s) = h$ because $c + s = 167/145$ and

$$390485 \cdot 167 = 65210995 = 449731 \cdot 145.$$

The rotated square (1.3) is the square R of the layout $\mathcal{L}(a, h, \theta)$ with $\cos \theta = c$ and $\sin \theta = s$, where $\theta \in (0, \pi/2)$ because $c, s > 0$. By Proposition 2.6 it suffices to verify that the layout is admissible.

The eight inequalities (2.4) are the positivity of the numerators in

$$\begin{aligned} a - \frac{1}{4} &= \frac{440211}{N}, & \frac{1}{4} - h &= \frac{126921}{N}, \\ b - h &= \frac{117401}{N}, & \frac{3}{8} - b &= \frac{297846}{N}, \\ 1 - a - 2h &= \frac{390283}{N}, & 1 - a - 2b &= \frac{155481}{N}, \\ a + 2b - \frac{3}{4} &= \frac{421171}{N}, & h &= \frac{449731}{N}, \end{aligned}$$

and each numerator is positive. For the contact inequalities (2.5), the relevant points are

$$P = \left(\frac{1016863}{N}, \frac{1320633}{N} \right), \quad Q = \left(\frac{1172344}{N}, \frac{1710916}{N} \right), \quad (x_0, y_0) = \left(\frac{957415}{N}, \frac{1289745}{N} \right),$$

and $145N = 334458160$ is a common denominator of α and β at these points. Multiplying by $145N$,

$$\begin{aligned} 145N \alpha(P) &= 143 \cdot 1016863 + 24 \cdot 1320633 = 177106601, \\ 145N \alpha(V_0) &= 143 \cdot 957415 + 24 \cdot 1289745 + \frac{1}{145} \cdot 143 \cdot 24 \cdot 390485 \\ &= 136910345 + 30953880 + 9242376 = 177106601, \end{aligned}$$

where $143 \cdot 24 \cdot 390485 = 1340144520 = 145 \cdot 9242376$. In the same way,

$$\begin{aligned} 145N \beta(Q) &= -24 \cdot 1172344 + 143 \cdot 1710916 = 216524732, \\ 145N \beta(V_2) &= -24 \cdot 957415 + 143 \cdot 1289745 + \frac{1}{145} \cdot 143^2 \cdot 390485 \\ &= -22977960 + 184433535 + 55069157 = 216524732, \end{aligned}$$

where $143^2 \cdot 390485 = 7985027765 = 145 \cdot 55069157$. Both contact inequalities therefore hold with equality, and the layout is admissible. By Lemma 2.5 the total side length is

$$3 + a + 2h + r = 3 + \frac{1016863 + 899462 + 390485}{N} = 3 + \frac{2306810}{2306608} = 4 + \frac{202}{2306608} = 4 + \frac{101}{1153304}. \quad \square$$

The two identities say that P lies on the line through the lower-left edge of R and Q on the line through its upper-left edge, so the packing has thirty-six pairs of squares in contact and the rotated square touches B_2 , D , C_{01} , E_0 and F_0 . The parameters in (1.2) are the solution of the linear system (1.4) at $\cos \theta = 143/145$, and Section 4 shows that every angle gives such a solution. Among the rotations by Pythagorean angles $\tan(\theta/2) = 1/n$ with $10 \leq n \leq 20$ the choice $n = 12$ gives the smallest denominators (Table 2), which is why it is the packing of Theorem A.

4. THE CONTACT FAMILY

Throughout this section let $\theta \in [0, \pi/2]$, $c = \cos \theta$, $s = \sin \theta$, $q = c + s$ and $t = \tan(\theta/2)$, so that

$$c = \frac{1 - t^2}{1 + t^2}, \quad s = \frac{2t}{1 + t^2}, \quad q = \frac{1 + 2t - t^2}{1 + t^2}, \quad 0 \leq t \leq 1. \quad (4.1)$$

4.1. The linear system.

Proposition 4.1 (The contact system). *In terms of (a, h, r) the contact equations (1.4) are*

$$\begin{pmatrix} c + 4s & 3c & -cs \\ 2s + 3c & -(5s + 2c) & -c^2 \\ 0 & -1 & c + s \end{pmatrix} \begin{pmatrix} a \\ h \\ r \end{pmatrix} = \begin{pmatrix} c + \frac{7}{4}s \\ \frac{3}{4}c \\ 0 \end{pmatrix}. \quad (4.2)$$

The determinant of the matrix is $-\Delta$ with $\Delta = 12c^3 + 31c^2s + 37cs^2 + 20s^3 > 0$, and the unique solution is (1.6). In the variable t ,

$$\begin{aligned} a &= \frac{(1 + 2t - t^2)(21 + 68t + 98t^2 - 68t^3 + 21t^4)}{8D(t)}, \\ h &= \frac{(1 + 2t - t^2)(9 + 34t + 38t^2 - 34t^3 + 9t^4)}{8D(t)}, \quad r = \frac{(1 + t^2)(9 + 34t + 38t^2 - 34t^3 + 9t^4)}{8D(t)}, \end{aligned} \quad (4.3)$$

with $D(t)$ as in (1.5), and $D(t) > 0$ for $0 \leq t \leq 1$.

Proof. By Lemma 2.4, the first equation of (1.4) is $ca + s(3a - 3/4) = c(1 - 3h) + s(1 - a) + crs$, which rearranges to $a(c + 4s) + 3ch - csr = c + 7s/4$. The second is $-s(1 - 2a + 2h) + c(1/4 + 2a - 2h) = -s(1 - 3h) + c(1 - a) + rc^2$, which rearranges to $a(2s + 3c) - h(5s + 2c) - c^2r = 3c/4$. The third is $-h + (c + s)r = 0$. Expansion along the last row gives the determinant $M_{32} + (c + s)M_{33}$, where

$$\begin{aligned} M_{32} &= -c^2(c + 4s) + cs(2s + 3c) = -c^3 - c^2s + 2cs^2, \\ M_{33} &= -(c + 4s)(5s + 2c) - 3c(2s + 3c) = -(11c^2 + 19cs + 20s^2), \end{aligned}$$

and the sum is $-\Delta$. Since $c, s \geq 0$ and $c^2 + s^2 = 1$, $\Delta \geq 12c^3 + 20s^3 > 0$. Substituting (1.6) into (4.2) verifies the three equations, and (4.3) follows from (4.1), because $4\Delta = 8D(t)/(1 + t^2)^3$. The polynomial D is positive on $[0, 1]$ by Lemma 4.2 below. $\square$

4.2. Admissibility along the family.

Lemma 4.2 (Positive polynomials). *For the solution (4.3) and $0 \leq t \leq 1$, the following identities hold, and every right-hand side is positive on $[0, 1]$:*

$$\begin{aligned} D(t) &= 6(1 - t^6) + 31t(1 + t^4) + 56t^2(1 - t^2) + 18t^3, \\ 8D(t)(a - \tfrac{1}{4}) &= 9(1 - t^6) + 48t(1 + t^4) + 101t^2(1 - t^2) + 24t^3, \\ 8D(t)(\tfrac{1}{4} - h) &= 3(1 - t^6) + 10t(1 + t^4) + 15t^2(1 - t^2) + 28t^3, \\ 8D(t)(b - h) &= (1 + 2t - t^2)(3 + 22t^2 + 3t^4), \\ 8D(t)(\tfrac{3}{8} - b) &= 6(1 - t^6) + 35t(1 + t^4) + 52t^2(1 - t^2) + 2t^3, \\ 8D(t)(1 - a - 2h) &= 9(1 - t^6) + 34t(1 + t^4) + 41t^2(1 - t^2) + 68t^3, \\ 8D(t)(1 - a - 2b) &= 3(1 - t^6) + 2t + 20t(1 - t^2) + 3t^2(1 - t^2) + 22t^5, \\ 8D(t)(a + 2b - \tfrac{3}{4}) &= 9(1 - t^6) + 40t(1 + t^4) + 109t^2(1 - t^2) + 56t^3, \\ 8D(t)h &= (1 + 2t - t^2)(9 + 34t(1 - t^2) + 38t^2 + 9t^4). \end{aligned}$$

Proof. The first line is (1.5) rearranged, and the others follow from (4.3) by expansion. In each of the seven unfactorised lines every term is nonnegative on $[0, 1]$, the first term is positive for $t < 1$, and the second term is positive at $t = 1$. In the two factorised lines, $1 + 2t - t^2 \geq 1 + t > 0$ and the second factor is a sum of terms nonnegative on $[0, 1]$ with positive constant term. $\square$

Proposition 4.3 (Every angle gives a packing). *For every $\theta \in [0, \pi/2]$ the layout $\mathcal{L}(a, h, \theta)$ with the parameters (1.6) is admissible. Its seventeen squares lie in the unit square, have pairwise disjoint interiors, and have total side length $4 + E(t)$, where*

$$E(t) = \frac{t^2(3 - 34t + 44t^2 - 34t^3 + 9t^4)}{4D(t)}. \quad (4.4)$$

Proof. The eight inequalities (2.4) are the positivity statements of Lemma 4.2, because $D(t) > 0$. The contact inequalities (2.5) hold with equality by Proposition 4.1. Proposition 2.6 applies, and by (4.3)

$$8D(t)(a + 2h + r - 1) = 2t^2(3 - 34t + 44t^2 - 34t^3 + 9t^4),$$

which is the total side length $3 + a + 2h + r = 4 + E(t)$. $\square$

The member at $t = 0$ has $(a, h, r) = (7/16, 3/16, 3/16)$, so its squares have sides $1/4, 3/16, 9/32, 1/4, 1/4, 3/16$ and $3/16$, all aligned, and its total side length is exactly four.

4.3. The excess.

Lemma 4.4 (The sign of the excess). *The quartic $Q(t) = 3 - 34t + 44t^2 - 34t^3 + 9t^4$ is strictly decreasing on $[0, 1]$, with $Q(0) = 3$ and $Q(1) = -12$. It has exactly one zero t_1 in $[0, 1]$, $t_1 > 1/10$, and $E(t) > 0$ exactly for $0 < t < t_1$.*

Proof. For $0 \leq t \leq 1$, $Q'(t) = -34 + t(88 - 102t + 36t^2) \leq -34 + t(88 - 66t) \leq -34 + 88^2/264 < 0$, because $36t^2 \leq 36t$ and $88t - 66t^2$ is at most $88^2/(4 \cdot 66)$. The value $Q(1/10) = 69/10000$ is positive, so $t_1 > 1/10$. Since $D > 0$ on $[0, 1]$, the sign of $E(t)$ for $t > 0$ is the sign of $Q(t)$. $\square$

Lemma 4.5 (The derivative of the excess). *With P as in (1.8),*

$$E'(t) = \frac{tP(t)}{4D(t)^2} \quad (0 \leq t \leq 1).$$

Proof. Write $E = N/(4D)$ with $N = t^2Q$. Then $4D^2E' = N'D - ND'$, and expansion of the right-hand side gives $tP(t)$. $\square$

Lemma 4.6 (Exact computation). *The polynomial P of (1.8) is irreducible over $\mathbb{Q}$. It has exactly three real zeros, at $-2.53357\dots$, at $t^* = 0.0620925223512813\dots$ and at $1.94278\dots$, and $P(0) = 36$, $P(1/10) < 0$ and $P(1) = -1776$, so $P > 0$ on $[0, t^*)$ and $P < 0$ on $(t^*, 1]$.*

Proof. The irreducibility is a factorisation over $\mathbb{Z}$ and the zero count is a Sturm sequence, both carried out in exact arithmetic by the accompanying program (Appendix A). The three values are $P(0) = 36$, $10^9P(1/10) = -25222871325$ and $P(1) = -1776$. $\square$

Proof of Theorem B. Proposition 4.1 gives the solution and Proposition 4.3 the packing with total side length $4 + E(t)$. Lemma 4.4 determines the sign of E . By Lemmas 4.5 and 4.6, E increases on $[0, t^*]$ and decreases on $[t^*, 1]$, so its maximum on $[0, 1]$ is $\varepsilon^* = E(t^*)$, attained only at t^* , and $t^* < 1/10 < t_1$ by Lemmas 4.4 and 4.6. The value of ε^* follows from the isolation of t^* in Appendix A. The statements about the field generated by t^* , $\cos \theta^*$ and ε^* are Proposition 4.7, and $f(17) \geq 4 + \varepsilon^*$ is the member at t^* . $\square$

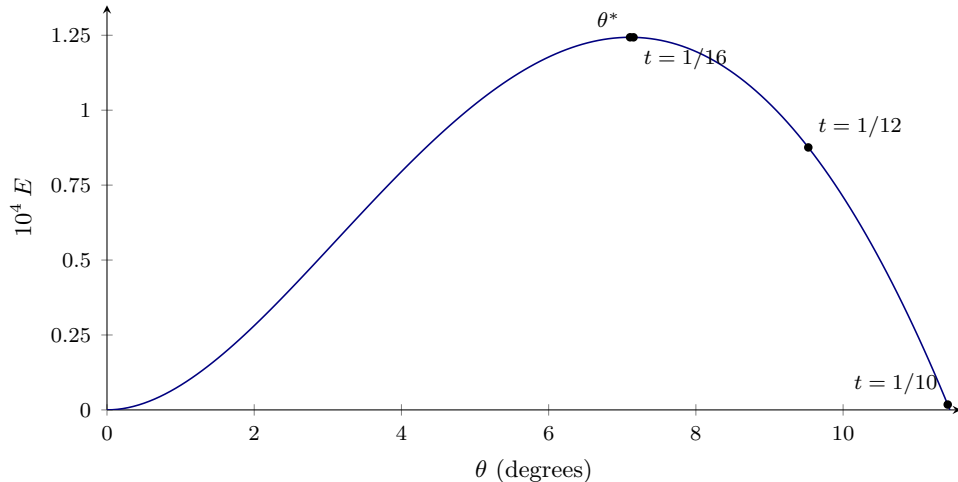


FIGURE 2. The excess E along the contact family as a function of the angle θ , in units of 10^{-4} . The marked points are the maximum ε^* at $\theta^* = 7.106^\circ$, Silverstein's packing at $t = 1/16$, the packing of Theorem A at $t = 1/12$, and the member at $t = 1/10$. The excess vanishes at $\theta_1 = 11.451^\circ$.

n	c	s	θ	denominator	$E(1/n)$	$10^4 E(1/n)$
10	99/101	20/101	11.421°	77 381 632	69/38690816	0.0178
11	60/61	11/61	10.389°	16 459 360	907/16459360	0.5511
12	143/145	24/145	9.527°	2 306 608	101/1153304	0.8757
13	84/85	13/85	8.797°	42 100 832	4497/42100832	1.0681
14	195/197	28/197	8.171°	512 319 200	30109/256159600	1.1754
15	112/113	15/113	7.628°	31 589 328	3877/31589328	1.2273
16	255/257	32/257	7.153°	1 095 192 400	68073/547596200	1.2431
17	144/145	17/145	6.733°	193 590 928	23917/193590928	1.2354
18	323/325	36/325	6.360°	23 106 752	1401/11553376	1.2126
19	180/181	19/181	6.026°	366 436 160	43251/366436160	1.1803
20	399/401	40/401	5.725°	3 938 257 712	224929/1969128856	1.1423

TABLE 2. The members of the contact family at the rational angles $\tan(\theta/2) = 1/n$, $10 \leq n \leq 20$: the rotation (c, s) , the angle, the common denominator of a , h and r , and the excess $E(1/n)$ over four. Every row is a packing of seventeen squares with total side length $4 + E(1/n)$.

4.4. The field of the optimum.

Proposition 4.7 (Galois group of the optimum). *Let K be the splitting field of P over $\mathbb{Q}$. Then $\text{Gal}(K/\mathbb{Q}) \cong S_9$. The field $F = \mathbb{Q}(t^*)$ has degree nine, has no proper intermediate field, and satisfies $F = \mathbb{Q}(\cos \theta^*) = \mathbb{Q}(\varepsilon^*)$. None of t^* , $\cos \theta^*$ and ε^* lies in a radical extension of $\mathbb{Q}$.*

Proof. Let $G = \text{Gal}(K/\mathbb{Q})$, viewed as a permutation group of the nine zeros of P . The monic polynomial $g(X) = 75^8 P(X/75)$ has integer coefficients, the same splitting field, and for every prime p not dividing 75 the same factorisation pattern modulo p as P , since the substitution $X \mapsto 75X$ is invertible modulo p . The discriminant of P is $-2^{40} \cdot 3^7 \cdot 5^6 \cdot 31^2 \cdot 97 \cdot 163 \cdot 443 \cdot 65381 \cdot 77003 \cdot 811771 \cdot 6857927$, and $\text{disc}(g) = 75^{56} \text{disc}(P)$, so the prime 23 divides neither. Modulo 23 the polynomial P is squarefree with irreducible factors of degrees 5, 2, 1 and 1 (Appendix A). Dedekind's theorem [2, Chapter 13] gives an element $\tau \in G$ of cycle type $(5, 2, 1, 1)$. Then τ^2 is a five-cycle and τ^5 is a transposition.

The group G is transitive, because P is irreducible. If G were imprimitive, its blocks would have size three, and the five-cycle τ^2 would permute the three blocks by a permutation of order dividing five, so it would fix each block and act on each block of size three by a permutation of order dividing five, that is, trivially, which contradicts that τ^2 moves five points. So G is primitive, and a primitive permutation group containing a transposition is the full symmetric group [3, Theorem 3.3A], [10, Theorem 13.3]. Therefore $G \cong S_9$.

The field $F = \mathbb{Q}(t^*)$ is the fixed field of the stabiliser of the root t^* , which is isomorphic to S_8 . A subgroup of S_9 containing a point stabiliser properly is transitive, so its order is at least $9 \cdot 8!$ and it is S_9 . Therefore F has no proper intermediate field, and every element of $F \setminus \mathbb{Q}$ generates F . The numbers $\cos \theta^* = (1 - t^{*2})/(1 + t^{*2})$ and $\varepsilon^* = E(t^*)$ lie in F , and neither is rational, since a rational value would give a nonzero polynomial of degree at most six over $\mathbb{Q}$ vanishing at t^* , whose degree is nine. Therefore $F = \mathbb{Q}(\cos \theta^*) = \mathbb{Q}(\varepsilon^*)$.

If one of these numbers lay in a radical extension of $\mathbb{Q}$, then F would lie in the Galois closure of a radical extension, whose Galois group is solvable in characteristic zero [2, Chapter 8], and $G \cong S_9$ would be a quotient of a solvable group. Since A_9 is simple and nonabelian, S_9 is not solvable, and no such extension exists. $\square$

4.5. The contacts are forced.

Proposition 4.8 (The two contacts maximise the total for a fixed angle). *Let $\theta \in (0, \pi/2]$, and let (a, h, r) be the member (1.6). If $\mathcal{L}(a', h', \theta)$ is admissible, then*

$$3 + a' + 2h' + r' \leq 3 + a + 2h + r, \quad r' = h'/(c + s),$$

with equality only for $(a', h') = (a, h)$.

Proof. For this θ the total side length is $3 + a' + h'(2 + 1/q)$, a linear function of (a', h') with gradient $u = (1, 2 + 1/q)$. The contact inequalities (2.5) are, by the proof of Proposition 4.1 with $r' = h'/q$,

$$n_1 \cdot (a', h') \leq c + \frac{7}{4}s, \quad n_2 \cdot (a', h') \leq -\frac{3}{4}c,$$

where $n_1 = (c + 4s, 3c - cs/q)$ and $n_2 = (-(2s + 3c), 5s + 2c + c^2/q)$, and the member (a, h) satisfies both with equality. Define

$$\lambda_1 = \frac{(1 + t^2)(6 + 19t + 9t^2 - 15t^3 + 3t^4)}{D(t)}, \quad \lambda_2 = \frac{t(3 - t)(1 + t^2)(4 + 7t + t^2)}{D(t)}.$$

Expansion with (4.1) gives $u = \lambda_1 n_1 + \lambda_2 n_2$, and both multipliers are positive for $0 < t \leq 1$, since $6 + 19t - 15t^3 \geq 6 + 4t$ and $D > 0$. For an admissible (a', h') define $\delta = (a', h') - (a, h)$. Then $n_1 \cdot \delta \leq 0$ and $n_2 \cdot \delta \leq 0$, so $u \cdot \delta = \lambda_1 n_1 \cdot \delta + \lambda_2 n_2 \cdot \delta \leq 0$, with equality only if $n_1 \cdot \delta = n_2 \cdot \delta = 0$. The vectors n_1 and n_2 are linearly independent, because their determinant $(c + 4s)(5s + 2c + c^2/q) + (2s + 3c)(3c - cs/q)$ is a sum of nonnegative terms whose first term is positive. So equality forces $\delta = 0$. $\square$

At $\theta = 0$ the second multiplier vanishes and the second contact carries no cost, which is the degenerate aligned member. For every positive angle both contacts are active at the optimum of the layout, and the family of Theorem B is the locus of these optima.

5. EVERY $k \geq 4$

Lemma 5.1 (Grid extension). *Let $1 \leq m \leq k$ be integers, and suppose that n squares with pairwise disjoint interiors lie in the unit square with total side length $m + \delta$. Then $n + k^2 - m^2$ squares with pairwise disjoint interiors lie in the unit square with total side length $k + m\delta/k$.*

Proof. Divide the unit square into the grid of k^2 squares of side $1/k$, and let $S = [0, m/k]^2$ be the union of the m^2 grid squares in the lower-left corner. Replace those m^2 grid squares by the images of the n given squares under the map $(x, y) \mapsto (m/k)(x, y)$. The images lie in S , have pairwise disjoint interiors, and have total side length $(m/k)(m + \delta)$. Every remaining grid square has its interior outside S . The new configuration therefore consists of $n + k^2 - m^2$ squares with pairwise disjoint interiors, and its total side length is

$$\frac{m(m + \delta)}{k} + \frac{k^2 - m^2}{k} = k + \frac{m\delta}{k}. \quad \square$$

Proof of Theorem C. Apply Lemma 5.1 with $m = 4$ and $n = 17$ to the packing of Theorem B at $t = t^*$, whose total side length is $4 + \varepsilon^*$. The result is a packing of $k^2 + 1$ squares with total side length $k + 4\varepsilon^*/k$, so $f(k^2 + 1) \geq k + 4\varepsilon^*/k$. The packing of Theorem A gives in the same way $f(k^2 + 1) \geq k + 4 \cdot 101/(1153304k) = k + 101/(288326k)$, and $4\varepsilon^* > 101/288326$ because $\varepsilon^* > 101/1153304 = E(1/12)$ by Theorem B. $\square$

Corollary 5.2 (Monotonicity and divergence). *For integers $1 \leq m \leq k$,*

$$k(f(k^2 + 1) - k) \geq m(f(m^2 + 1) - m),$$

so $k(f(k^2 + 1) - k)$ is a nondecreasing function of k , and the series $\sum_{k \geq 1} (f(k^2 + 1) - k)$ diverges.

Proof. Apply [Lemma 5.1](#) with $n = m^2 + 1$ to packings of $m^2 + 1$ squares whose total side length approaches $f(m^2 + 1)$, and take the supremum: $f(k^2 + 1) \geq k + m(f(m^2 + 1) - m)/k$. The terms of the series are at least $4\varepsilon^*/k$ for $k \geq 4$ by [Theorem C](#). $\square$

The monotonicity is the observation of Singh (arXiv:2601.22163, 2026), from which he derived that the conjecture $f(k^2 + 1) = k$ holds for every k if and only if the series converges. The divergence is the quantitative form of its failure.

Proof of Corollary D. For an integer c let $\Pi(c)$ be the statement that $f(k^2 + 2c + 1) = k + c/k$ for every integer $k \geq |c|$. Praton [8] proved that $\Pi(c)$ implies $\Pi(c - 1)$ and $\Pi(c + 1)$, so the statements $\Pi(c)$ are all equivalent. The statement $\Pi(0)$ is false by [Theorem A](#), so $\Pi(c)$ is false for every c . Since $f(k^2 + 2c + 1) \geq k + c/k$ for $k \geq |c|$ by (1.1) and by $f(j^2) = j$ in the cases $c = \pm k$, the failure of $\Pi(c)$ is an integer $k \geq |c|$ with $f(k^2 + 2c + 1) > k + c/k$. $\square$

Remark 5.3. [Corollary D](#) does not locate k when $c \neq 0$. The grid extension adds exactly one square to a configuration, so it does not combine with the constructions behind (1.1) to raise the total of $k^2 + 2c + 1$ squares above $k + c/k$ for a prescribed k . The cases $k \leq 3$ of the original question are also untouched: $f(2) = 1$, $f(5) = 2$, and whether $f(10) = 3$ remains open.

APPENDIX A. EXACT COMPUTATIONS

Proposition A.1 (Root isolation). *The quartic $Q(t) = 3 - 34t + 44t^2 - 34t^3 + 9t^4$ has exactly two real zeros, none in $[0, 1/10]$ and one in $[1/10, 1]$, namely*

$$t_1 = 0.10026361082159358364937941908\dots, \quad \theta_1 = 2 \arctan t_1 = 11.45109398384791363\dots^\circ,$$

and the other at $2.35870684\dots$. The polynomial P of (1.8) has exactly three real zeros, one in $[0, 1/10]$ and none in $[1/10, 1]$, namely

$$t^* = 0.06209252235128130618810292551\dots, \quad \theta^* = 2 \arctan t^* = 7.10615576032722724823\dots^\circ,$$

and the others at $-2.53357738\dots$ and $1.94278108\dots$. Consequently

$$\varepsilon^* = E(t^*) = 0.00012432478481177123742209\dots, \quad \cos \theta^* = 0.9923186526275100776\dots,$$

and $4\varepsilon^* = 0.00049729913924708\dots > 1/2011$.

Proof. The zero counts on the stated intervals are Sturm sequences evaluated at the rational endpoints, and the digits are obtained by bisection in exact rational arithmetic, both by the accompanying program. The counts for Q and the sign of P at 0, $1/10$ and 1 are also those of [Lemmas 4.4](#) and [4.6](#). $\square$

Proposition A.2 (Galois data). *The polynomial P is irreducible over $\mathbb{Q}$, and*

$$\text{disc}(P) = -2^{40} \cdot 3^7 \cdot 5^6 \cdot 31^2 \cdot 97 \cdot 163 \cdot 443 \cdot 65381 \cdot 77003 \cdot 811771 \cdot 6857927.$$

Modulo 23,

$$P(t) \equiv 6(t + 2)(t - 10)(t^2 - 8t + 2)(t^5 + 5t^4 + 11t^3 + 6t^2 - 2t + 1),$$

where the quadratic and the quintic are irreducible over the field with 23 elements.

Proof. The irreducibility over $\mathbb{Q}$ is a factorisation over $\mathbb{Z}$, and the factorisation modulo 23 is computed twice, by a computer algebra system and by an independent distinct-degree factorisation written for this purpose, which also checks that P is squarefree modulo 23. The quadratic is irreducible because its discriminant $56 \equiv 10$ is not a square modulo 23, and the irreducibility of the quintic is part of the distinct-degree computation. The discriminant is computed exactly and factorised. $\square$

Proposition A.3 (The minimal polynomial of the optimal excess). *The number ε^* is the unique positive zero of the irreducible polynomial*

$$\begin{aligned} m(x) = & 60108126177280000 x^9 + 166935576571904000 x^8 + 201158809349760000 x^7 \\ & + 129033432231173120 x^6 + 44759877570060992 x^5 + 7765816854698688 x^4 \\ & + 557636251906144 x^3 + 11953652807976 x^2 + 164724750117 x - 20665206, \end{aligned}$$

whose other real zeros are $-0.29128827\dots$ and $-0.11593976\dots$.

Proof. The resultant of $t^2Q(t) - 4xD(t)$ and $P(t)$ with respect to t is $532176772792320000 m(x)$, so $m(\varepsilon^*) = 0$. The polynomial m is irreducible over $\mathbb{Q}$, in agreement with $[\mathbb{Q}(\varepsilon^*) : \mathbb{Q}] = 9$ in [Proposition 4.7](#), and its real zeros are isolated as in [Proposition A.1](#). $\square$

The accompanying programs verify the packing of [Theorem A](#) in exact rational arithmetic, by the separating-axis test and by half-plane clipping of every pair of squares, and repeat the verification for every member of the contact family at the angles $\tan(\theta/2) = p/q$ with $0 \leq p \leq q \leq 24$. They verify the identities of [Proposition 4.1](#), [Lemmas 4.2](#) and [4.5](#), and the multipliers of [Proposition 4.8](#) symbolically, and they carry out the computations of [Propositions A.1](#) to [A.3](#). Their inputs, arithmetic conventions and recorded outputs are provided with the source. The theorems are proved by the arguments above, and the finite computations enter only through [Lemma 4.6](#) and the propositions of this appendix.

REFERENCES

- [1] C. Campbell and W. Staton, A square-packing problem of Erdős, *Amer. Math. Monthly* **112** (2005), no. 2, 165–167.
- [2] D. A. Cox, *Galois theory*, 2nd ed., Pure and Applied Mathematics, John Wiley & Sons, Hoboken, NJ, 2012.
- [3] J. D. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics 163, Springer-Verlag, New York, 1996.
- [4] P. Erdős, Some problems in number theory, combinatorics and combinatorial geometry, *Math. Pannon.* **5** (1994), no. 2, 261–269.
- [5] P. Erdős and R. L. Graham, On packing squares with equal squares, *J. Combin. Theory Ser. A* **19** (1975), 119–123.
- [6] P. Erdős and A. Soifer, Squares in a square, *Geombinatorics* **4** (1995), no. 4, 110–114.
- [7] S. Halász, Packing a convex domain with similar convex domains, *J. Combin. Theory Ser. A* **37** (1984), 85–90.
- [8] I. Praton, Packing squares in a square, *Math. Mag.* **81** (2008), no. 5, 358–361.
- [9] W. Staton and B. Tyler, On the Erdős square-packing conjecture, *Geombinatorics* **17** (2007), no. 2, 88–94.
- [10] H. Wielandt, *Finite permutation groups*, translated from the German by R. Bercov, Academic Press, New York–London, 1964.

LONDON, UNITED KINGDOM

Email address: budden@pingyou.com