

THE FIBONACCI GEOMETRY OF ELEVEN SQUARES

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ABSTRACT. We give a reversible representation of Trump’s packing of eleven unit squares by eleven congruent square–rectangle cells on a marked flat torus. The contact equations have Fibonacci form $I + wQ$, where $Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $w = \sin \theta \cos \theta$. One marked point recovers every planar square, both active clearances and the enclosing side. We prove that two inequalities characterize feasibility in a two-parameter contact family. They force its unique minimum $T_* = 3.877083590\dots$ and the bound $T - T_* \geq \frac{1}{4}|u - u_*|$, where $u = \tan(\theta/2)$.

Let $\varphi = (1 + \sqrt{5})/2$. Among the quotients $R_n = \mathbb{Z}[\varphi]/(\varphi^n - 1)$ with $n \geq 3$, the group generated by $x \mapsto x + 1$ and $x \mapsto \varphi x$ acts freely and transitively on unordered pairs exactly when $n = 5$, with $R_5 \cong \mathbb{F}_{11}$. The reciprocal map $[\alpha, \beta] \mapsto \{\alpha \pm \beta^{-1}\}$ over $\mathbb{F}_{11}$ identifies the fifty-five two-element return orbits on the 121-cell cover with all fifty-five pairs of base cells. The torus bundle defined by $-Q^3$ and its cyclic covers recover the same finite modules and affine group. The normalized tori lie on a shortest closed modular geodesic of length $4 \log \varphi$. We prove that the minimizing side generates a degree-eight number field whose normal closure has the full symmetric Galois group S_8 . We also give an explicit rational counterexample to Erdős’s square-packing conjecture (Problem 106).

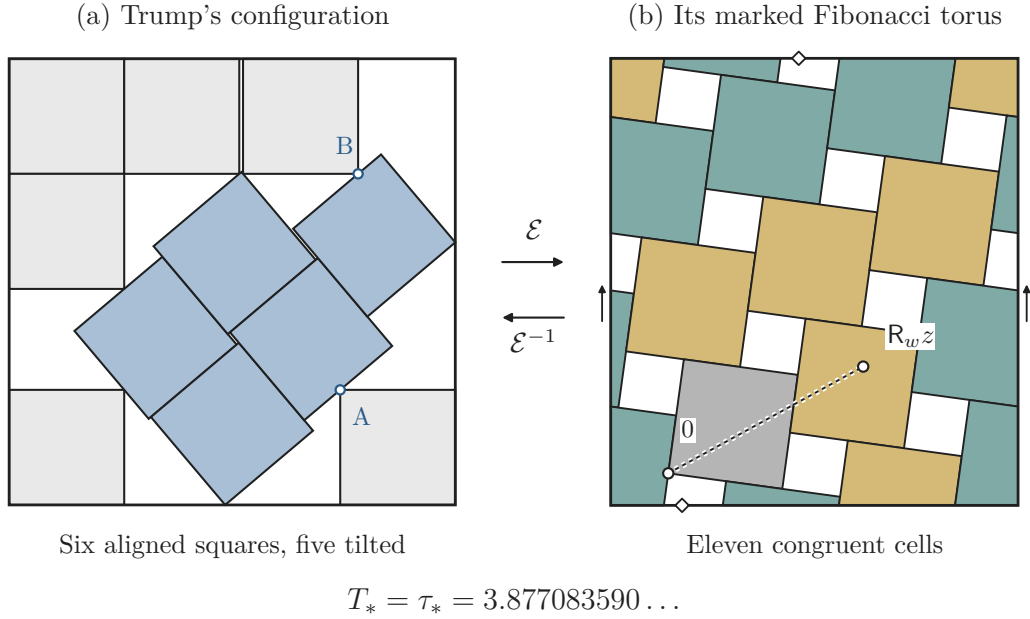


FIGURE 1. Trump’s packing and its marked Fibonacci torus of square–rectangle cells, at equal unit-square scale. The five tilted squares are blue, and the six aligned squares are light gray. The active contacts A and B , circled in dark blue, are enlarged in Figure 2. On the torus, the residue classes $H = \{1, 3, 4, 5, 9\}$ and $-H$ are gold and teal, respectively, and zero is dark gray. Opposite vertical edges are identified by the arrows, and the shifted identification of the horizontal edges is specified by the paired diamonds. The dotted segment joins the origin to $R_w z$, where z is the mark’s admissible lift and R_w rotates the first period to the horizontal. The bijection $\mathcal{E}$ preserves the enclosing side of the whole configuration, with minimum $T_* = \tau_*$.

1. A MARKED FIBONACCI TORUS RECOVERS THE ELEVEN-SQUARE PACKING

Trump's packing consists of six axis-parallel unit squares and five tilted unit squares. We represent it by eleven congruent square-rectangle cells on a marked flat torus (Figure 1). The mark recovers every planar square, both active clearances and the enclosing side.

The contact equations determine the Fibonacci matrix

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad Q^2 = Q + I,$$

in the geometric normalization of Lemma 3.1. The same matrix governs the eleven-cell quotient, the cyclic covers of a torus bundle, and an affine group that acts regularly on pairs of cells.

1.1. Two contacts force the family minimum. Let $u = \tan(\theta/2)$, where θ is the common tilt angle, and let T be the enclosing side. Section 2 defines a contact family on

$$\mathcal{R} = \left[\frac{9}{25}, \frac{37}{100} \right] \times \left[\frac{387}{100}, \frac{777}{200} \right].$$

Write $\mathcal{P}$ for its feasible members.

Theorem A (Two contacts determine the minimum). *A point of $\mathcal{R}$ is feasible if and only if two explicit signed clearances g_A and g_B are nonnegative. Their enclosing-side bounds $T_A(u)$ and $T_B(u)$ cross exactly once, at (u_*, T_*) . Every $(u, T) \in \mathcal{P}$ satisfies*

$$T - T_* \geq \frac{1}{4}|u - u_*|.$$

The clearances are analytic local coordinates at the minimum. Opening either contact has a strictly positive first-order cost in T .

The minimizing configuration is Trump's packing, with

$$T_* = 3.877083590022814177307\dots, \quad \theta_* = 40.18193729\dots^\circ.$$

Lemma 2.2, Theorem 2.4, and Proposition 2.5 prove Theorem A using the containment and separation bounds of Appendix A.

Define

$$c = \cos \theta, \quad s = \sin \theta, \quad w = cs, \quad M_w = I + wQ, \quad C_- = 4I - Q, \quad C_+ = 3I + Q.$$

The cell lattice is $M_w\mathbb{Z}^2$. Since $Q^5 - I = Q^3C_-$, its quotient by $M_wC_-\mathbb{Z}^2$ is the fifth-return quotient.

Theorem B (One marked point recovers the packing). *In fixed geometric coordinates, the contact equations have the symmetric form*

$$M_w z = \xi + \lambda e, \quad \lambda = s^2 g_B,$$

where ξ and e depend only on θ . The translates of

$$F_w = [0, 1]^2 \cup ([w, 1] \times [1, 1 + w])$$

by $M_w\mathbb{Z}^2$ tile the plane. Eleven such cells tile

$$\mathbb{T}_w^+ = \mathbb{R}^2 / (M_w C_- \mathbb{Z}^2).$$

Every admissible marked point on this torus has a unique admissible lift z . The map from $\mathcal{P}$ to the resulting marked family is a semialgebraic bijection. It preserves g_A , g_B and T .

The quarter turn $\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ carries this family isometrically to the conjugate family with period matrix $\text{adj}(M_w)C_+$. It conjugates the cell exchange by Q^2 to the exchange by Q^{-2} .

Theorems 3.2, 3.5 and 4.1, Corollary 3.3, and Proposition 3.4 prove Theorem B. The correspondence identifies whole configurations and their constraints. It does not identify individual planar squares with torus cells. For fixed w , different enclosing sides give different marks on the same torus (Corollary 3.6).

1.2. The cubic return yields eleven residues and fifty-five pairs. Let $\varphi = (1 + \sqrt{5})/2$ and $\mathcal{O} = \mathbb{Z}[\varphi]$. The basis $(1, \varphi)$ identifies Q with multiplication by φ . The cubic iterate of the contact equation has linear part $w^3 R$, where $R = -Q^3$.

Theorem C (The cubic return recovers both residue modules). *The universal abelian quotient on which R and Q^{-2} agree is*

$$\mathbb{Z}^2 / C_+ \mathbb{Z}^2 \cong \mathbb{F}_{11}.$$

Let A_n be the torsion subgroup of the first integral homology of the suspension of R^n . Then

$$A_n \cong \mathcal{O} / ((-\varphi^3)^n - 1).$$

Its eleven-primary subgroup is nonzero exactly when $5 \mid n$. The first such level is $n = 5$, and the first orientable such level is $n = 10$. Their eleven-primary subgroups are

$$A_5[11^\infty] \cong \mathbb{F}_{11}, \quad A_{10}[11^\infty] \cong \mathbb{F}_{11}^2.$$

In the residue coordinates $(\alpha, \beta) = (m + 4n, m + 8n)$, the degree-two covering induces projection $(\alpha, \beta) \mapsto \beta$ and transfer $\beta \mapsto (0, 2\beta)$.

Theorems 5.5 and 5.7 and Proposition 5.6 prove Theorem C.

For $n \geq 3$, define $R_n = \mathcal{O} / (\varphi^n - 1)$. Let G_n be the permutation group generated by $x \mapsto x + 1$ and $x \mapsto \varphi x$.

Theorem D (The unique return quotient with a regular pair action). *The action of G_n on unordered pairs of distinct elements of R_n is transitive exactly for $n = 3, 4, 5$. It is free and transitive exactly for $n = 5$, where $R_5 \cong \mathbb{F}_{11}$ and $|G_5| = 55$.*

On $\mathcal{O} / (11) \cong \mathbb{F}_{11}^2$, the fifth return acts by $(\alpha, \beta) \mapsto (\alpha, -\beta)$. The map

$$[\alpha, \beta] \mapsto \{\alpha + \beta^{-1}, \alpha - \beta^{-1}\}, \quad \beta \neq 0,$$

is an equivariant bijection from its fifty-five two-element orbits to the fifty-five unordered pairs in $\mathbb{F}_{11}$. The five orbits over each α give a matching of all ten other labels. These eleven matchings partition all pairs.

The fundamental group of the cubic suspension maps onto G_5 . The action of G_5 on the eleven-cell torus consists of piecewise translations that preserve each cell's square and rectangle.

Theorems 6.1 and 6.4, Corollaries 6.3 and 6.5, and Proposition 6.6 prove the pair classification and its geometric realization. Pair regularity is not assumed as a constraint on the planar contacts.

1.3. The minimizing side generates a degree-eight field.

Theorem E (The minimizing side is not expressible in radicals). *The side T_* has degree eight over $\mathbb{Q}$. Its minimal polynomial has Galois group S_8 and exactly two real roots, one negative and the other T_* . The field $F = \mathbb{Q}(T_*)$ has no proper intermediate field over $\mathbb{Q}$, and*

$$F = \mathbb{Q}(u_*) = \mathbb{Q}(\cos \theta_*) = \mathbb{Q}(w_*) = \mathbb{Q}(\gamma_o),$$

where γ_o is the density of the periodic extension in Appendix B. In particular, T_* is not expressible in radicals and $\sqrt{5} \notin F$.

Theorem 8.1 and Corollary 8.2 prove Theorem E using the polynomial computations of Appendix C. The area-normalized tori lie on a shortest closed modular geodesic, with the square member at $w = 1/3$ (Theorem 7.1). The original eleven-square motif also extends periodically with a smaller fundamental area than its enclosing square (Corollary 3.9). Appendix B gives the construction and proves nonoverlap.

1.4. The contact-family analysis extends the known tilt equation. Let $s(n)$ be the least enclosing side for n unit squares with pairwise disjoint interiors and arbitrary orientations. Published exact values include $n \leq 10$ [9, 12, 15, 18], $n = m^2, m^2 - 1$ [13, 16], and $n = 13, 46$ [4].

Trump found the configuration in 1979, and Gardner published it in [10, pp. 299–300]. Friedman surveys its history [9]. Trump's note [19] records his tilt equation and the later numerical work of Gensane and Ryckelynck [11], including the equation

$$5c^8 - 12c^7 + 8c^6 + 2c^5 - c^4 - 2c + 1 = 0$$

for $c = \cos \theta_*$. Substituting $c = (1 - u^2)/(1 + u^2)$ turns its left-hand side into $P(u)P(-u)/(1 + u^2)^8$, where P is the polynomial in Theorem 2.4. Thus the equation determining the tilt predates this paper.

A proposed proof of the unrestricted lower bound $s(11) \geq T_*$ is given in [1]. The two enclosing-side balances and their crossing identity appear in [2, Section 8.4], together with a quantitative tilt bound under specified projection orders. We construct the full two-parameter contact family and prove that these inequalities characterize feasibility. We sharpen the tilt bound and use the clearances as local coordinates.

2. TWO CONTACTS DETERMINE THE FAMILY MINIMUM

2.1. The tilt and enclosing side determine every square. Define

$$\mathcal{I} = \left[\frac{9}{25}, \frac{37}{100} \right], \quad \mathcal{J} = \left[\frac{387}{100}, \frac{777}{200} \right], \quad \mathcal{R} = \mathcal{I} \times \mathcal{J}.$$

For $(u, T) \in \mathcal{R}$ define

$$c = \frac{1 - u^2}{1 + u^2}, \quad s = \frac{2u}{1 + u^2}, \quad t = T - 3, \quad e_1 = (c, s), \quad e_2 = (-s, c). \quad (2.1)$$

Then $c^2 + s^2 = 1$. The five rotated squares have common tilt $\theta = 2 \arctan u$, with $c = \cos \theta$ and $s = \sin \theta$. Define

$$\rho = 1 - ct, \quad \eta = \frac{2c - 1 - c^2 t}{s}, \quad x_0 = \frac{c + 2 - s - st}{c}, \quad \zeta = \frac{T - 1}{s} - \rho - \frac{(3 + \eta)c}{s}. \quad (2.2)$$

Let $U(a, b) = [a, a + 1] \times [b, b + 1]$ and define

$$G(x, y) = (1, 1) + xe_1 + (y - \rho)e_2.$$

Table 1 lists eleven unit squares $K_0, \dots, K_{10}$ as functions of (u, T) .

i	K_i	i	K_i
0	$U(0, 0)$	6	$G(U(0, 0))$
1	$U(T - 1, 0)$	7	$G(U(\eta, -1))$
2	$U(x_0, T - 1)$	8	$G(U(1, c - s))$
3	$U(0, T - 1)$	9	$G(U(\eta + 1, c - s - 1))$
4	$U(1, T - 1)$	10	$G(U(\eta + 2, -\zeta))$
5	$U(0, T - 2)$		

TABLE 1. The eleven unit squares of the contact family as functions of (u, T) , through (2.1) and (2.2).

The squares K_0 and K_1 occupy the bottom corners of $[0, T]^2$. The squares K_3, K_4 and K_5 form the top-left L-shaped block, while K_2 lies on the top side. The left and top sides of K_6 pass through the corners $(1, 1)$ of K_0 and $(1, T - 2)$ of K_5 , since $G(0, \rho) = (1, 1)$ and $G(st, 1) = (1, T - 2)$.

The other four rotated squares are translates of K_6 in its own frame. The square K_7 lies below K_6 and touches the bottom side of $[0, T]^2$. The square K_8 lies to the right of K_6 and touches the corners $(2, T - 1)$ of K_4 and $(x_0, T - 1)$ of K_2 . The square K_9 lies below K_8 , and

K_{10} touches the right side of $[0, T]^2$. These four contacts define η , x_0 and ζ in (2.2). The bounds in Appendix A place each contact on the stated side. Let $\mathcal{P}$ be the set of $(u, T) \in \mathcal{R}$ for which the eleven squares lie in $[0, T]^2$ and have pairwise disjoint interiors. Each member carries the frame and labels of Table 1.

2.2. Two clearances characterize feasibility. For a unit vector n define

$$\ell_i(n) = \min_{x \in K_i} n \cdot x, \quad r_i(n) = \max_{x \in K_i} n \cdot x.$$

The two signed clearances are

$$g_A = \ell_9(e_2) - r_1(e_2), \quad g_B = \ell_{10}(-e_2) - r_2(-e_2), \quad (2.3)$$

the signed distances between the corresponding supporting lines. Define

$$D = 1 + cs - c^2s^2, \quad T_A = 3 + \frac{2-c}{c+s}, \quad T_B = 3 + \frac{c+c^3-1+s(2-c+c^2-c^3)}{D}. \quad (2.4)$$

Lemma 2.1 (Separating axis). *Let K and K' be convex polygons with nonempty interiors. Their interiors are disjoint if and only if some unit vector n parallel to a side normal of K or of K' satisfies $\min_{x \in K'} n \cdot x \geq \max_{x \in K} n \cdot x$.*

Proof. The interiors of K and K' meet if and only if the origin lies in the interior of the convex polygon $K' - K = \{y - x : y \in K', x \in K\}$. For a convex polygon A , let $\mathcal{F}_A(n)$ be its face maximizing $n \cdot x$. Then

$$\mathcal{F}_{K'-K}(n) = \mathcal{F}_{K'}(n) - \mathcal{F}_K(-n).$$

If the face on the left is a side, at least one face on the right is a side. Both faces on the right lie in lines perpendicular to n , so every side of $K' - K$ is parallel to a side of K or of K' .

The interior of a convex polygon is the intersection of the open half-planes bounded by its sides. Thus the origin lies outside the interior of $K' - K$ if and only if some side with outward unit normal n satisfies

$$0 \geq \max_{K'-K} n \cdot y = \max_{K'} n \cdot y - \min_K n \cdot x.$$

Replacing n by $-n$ gives the asserted inequality. Conversely, that inequality separates the two interiors. $\square$

Lemma 2.2 (Two clearances characterize feasibility). *For every $(u, T) \in \mathcal{R}$ the eleven squares lie in $[0, T]^2$, and*

$$g_A = (c+s)(T - T_A), \quad g_B = \frac{D}{cs^2}(T - T_B). \quad (2.5)$$

The eleven squares have pairwise disjoint interiors if and only if $g_A \geq 0$ and $g_B \geq 0$. Therefore

$$\mathcal{P} = \{(u, T) \in \mathcal{R} : T \geq \max\{T_A(u), T_B(u)\}\}. \quad (2.6)$$

Proof. Lemma A.2 proves containment. Every side of every square is normal to one of $e_x = (1, 0)$, $e_y = (0, 1)$, e_1 and e_2 , so Lemma 2.1 applies with the eight signed normals $\pm e_x, \pm e_y, \pm e_1, \pm e_2$. By Lemma A.3, the fifty-three pairs other than (1, 9) and (2, 10) have disjoint interiors throughout $\mathcal{R}$. By Lemma A.4, only e_2 can separate (1, 9), and only $-e_2$ can separate (2, 10). Their separation is therefore equivalent to $g_A \geq 0$ and $g_B \geq 0$, respectively. Lemma A.4 also gives

$$g_A = (c+s)t - (2-c), \quad g_B = c + \zeta - sx_0,$$

and substitution of (2.2) turns these into (2.5). The factors $c+s$ and $D/(cs^2)$ are positive on $\mathcal{R}$, which proves (2.6). $\square$

2.3. The contact bounds cross at a sharp minimum. Let $u_0 = 9/25$, $u_1 = 37/100$ and $M(u) = \max\{T_A(u), T_B(u)\}$, so that $\mathcal{I} = [u_0, u_1]$. Define

$$P(u) = 5u^8 - 10u^7 - 2u^6 + 14u^5 + 12u^4 - 6u^3 + 2u^2 + 2u - 1. \quad (2.7)$$

Lemma 2.3 (Uniform slope bounds). *On $\mathcal{I}$ the bounds satisfy $T'_A > 1/4$ and $T'_B < -1/4$, and $T_A - T_B$ has the sign of P .*

Proof. Let $E = (1 + u^2)^4 D$, which is positive. Differentiation and subtraction give

$$T_A - T_B = \frac{sP(u)}{D(c+s)(1+u^2)^4}, \quad T'_A = \frac{2(3u^2 + 4u - 1)}{(1 + 2u - u^2)^2}, \quad T'_B = -\frac{4u p_B(u)}{E^2}, \quad (2.8)$$

where

$$p_B(u) = 2 - 7u + 8u^2 + u^3(18 - 38u) + u^5(47 - 48u) + u^7(12 + 40u^3 + 2u^4 + u^6) - u^8(26 + 73u + 2u^4).$$

The numerator of T'_A is at least $2(3u_0^2 + 4u_0 - 1)$ and its denominator is at most $(1 + 2u_1)^2$. Their quotient exceeds $1/4$. The quadratic $2 - 7u + 8u^2$ decreases on $\mathcal{I}$. The factors $18 - 38u$ and $47 - 48u$ are positive there, as is the seventh-power group. Also $26 + 73u + 2u^4 < 57$, since $u < 2/5$. Dropping the seventh-power group gives

$$p_B(u) > \frac{1263}{2500} + u_0^3(18 - 38u_1) + u_0^5(47 - 48u_1) - 57 \left(\frac{2}{5}\right)^8 = \frac{809054901}{976562500} > \frac{4}{5}.$$

Since $D = 1 + cs - (cs)^2 \leq 5/4$, the quantity E is at most $(1 + u_1^2)^4 \cdot 5/4 < 21/10$, so

$$-T'_B > \frac{4u_0 \cdot (4/5)}{(21/10)^2} = \frac{64}{245} > \frac{1}{4}.$$

Every factor other than P in the first identity of (2.8) is positive, which gives the sign statement. $\square$

Theorem 2.4 (The sharp family minimum). *The polynomial P has exactly one zero u_* in $\mathcal{I}$. The unique member of $\mathcal{P}$ of least enclosing side is (u_*, T_*) , where*

$$T_* = T_A(u_*) = T_B(u_*) = \frac{6u_* + 4}{1 + 2u_* - u_*^2},$$

and every $(u, T) \in \mathcal{P}$ satisfies

$$T - T_* \geq \frac{1}{4} |u - u_*|. \quad (2.9)$$

The decimal expansions are

$$u_* = 0.3657693076046772933885 \dots, \quad T_* = 3.877083590022814177307 \dots$$

Proof. At the endpoints of $\mathcal{I}$,

$$P(u_0) = -\frac{772365104}{30517578125} < 0, \quad P(u_1) = \frac{37988917991321}{2 \cdot 10^{15}} > 0.$$

By Lemma 2.3, $T_A - T_B$ is strictly increasing on $\mathcal{I}$ and has the sign of P . It therefore vanishes at exactly one point u_* , the only zero of P in $\mathcal{I}$. Substitution of (2.1) into (2.4) gives $T_A(u) = (6u + 4)/(1 + 2u - u^2)$ and the endpoint bounds

$$3.873 < T_A(u_0) = \frac{275}{71} < T_A(u_1) = \frac{62200}{16031} < 3.880, \quad 3.875 < T_B(u_1) < T_B(u_0) < 3.879,$$

so the graph of M lies in $\mathcal{R}$. By Lemma 2.2, $(u, M(u)) \in \mathcal{P}$ for every $u \in \mathcal{I}$.

If $u > u_*$, integration of $T'_A > 1/4$ gives $M(u) \geq T_A(u) > T_* + (u - u_*)/4$. If $u < u_*$, integration of $T'_B < -1/4$ gives $M(u) \geq T_B(u) > T_* + (u_* - u)/4$. Every member (u, T) of $\mathcal{P}$ has $T \geq M(u)$. This proves (2.9) and the uniqueness of the minimizing member. Proposition A.6 supplies the decimal expansions. $\square$

Proposition 2.5 (Positive first-order cost at both contacts). *The map $(u, T) \mapsto (g_A, g_B)$ is a system of analytic local coordinates near (u_*, T_*) . Let*

$$a_* = c_* + s_*, \quad b_* = \frac{D_*}{c_* s_*^2}, \quad A' = T'_A(u_*), \quad B' = T'_B(u_*),$$

with c_* , s_* and D_* the values at u_* . In these coordinates

$$T = T_* + \frac{-B'}{a_*(A' - B')} g_A + \frac{A'}{b_*(A' - B')} g_B + O(g_A^2 + g_B^2),$$

and both linear coefficients are positive.

Proof. All functions involved are analytic near (u_*, T_*) . Since $T - T_A$ and $T - T_B$ vanish there, differentiating (2.5) at that point gives the Jacobian matrix

$$\begin{pmatrix} -a_* A' & a_* \\ -b_* B' & b_* \end{pmatrix}$$

with respect to (u, T) , whose determinant $a_* b_*(B' - A')$ is negative by Lemma 2.3. The analytic inverse function theorem applies, and inverting the displayed linear map gives the two coefficients, which are positive because $A' > 0 > B'$. $\square$

The contact equations in Section 1 identify (u_*, T_*) with Trump's configuration at tilt $\theta_* = 2 \arctan u_* = 40.18193729 \dots^\circ$. By Lemma A.5, this unique minimum persists under sufficiently small changes of the endpoints of $\mathcal{R}$ that retain (u_*, T_*) . Figure 2 shows the clearances above the minimum and the feasible region $\mathcal{P}$.

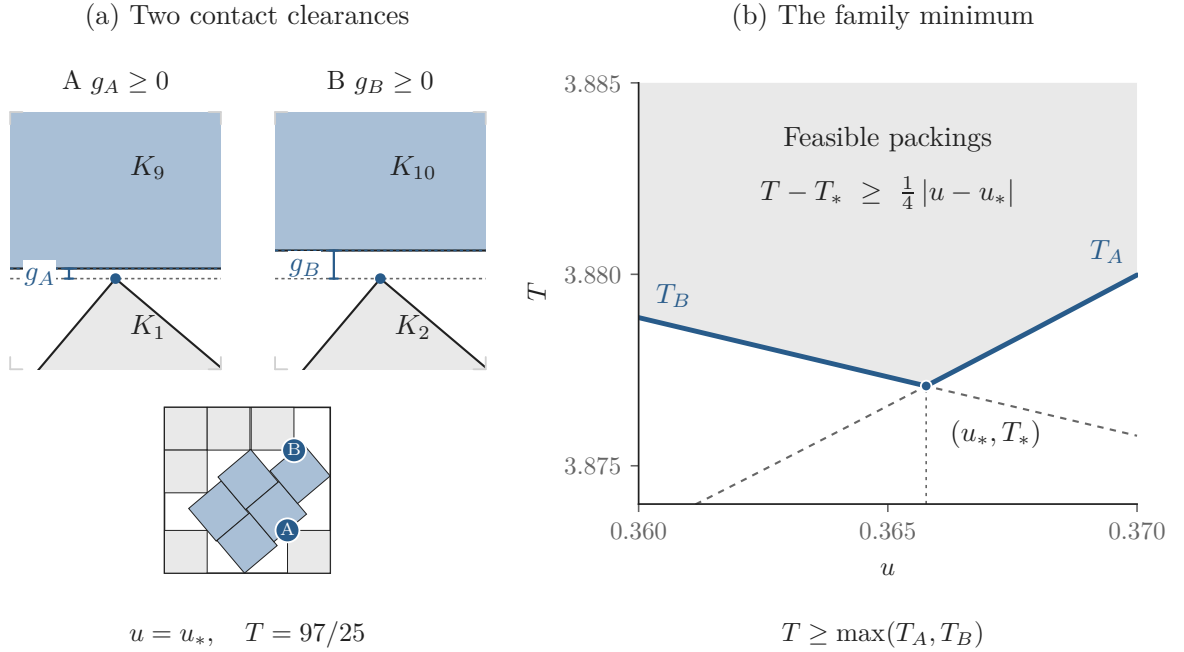


FIGURE 2. The two contact clearances and their enclosing-side bounds. The pairs $A = (K_1, K_9)$ and $B = (K_2, K_{10})$ are located in the small packing below the close-ups. Tilted squares are blue and aligned squares are light gray, as in Figure 1. At $u = u_*$ and $T = 97/25$, the close-ups use the frames (e_1, e_2) for A and $(-e_1, -e_2)$ for B , so both clearances are vertical. At right, $u = \tan(\theta/2)$ parametrizes the tilt and T is the enclosing side. The bounds $T_A(u)$ and $T_B(u)$ are the side lengths at which g_A and g_B vanish. The feasible region is shaded above the solid blue branches; inactive branches are dashed gray. Its minimum is the crossing of T_B and T_A .

3. THE CONTACT EQUATIONS DETERMINE A FIBONACCI LATTICE

3.1. Geometric coordinates determine a unique symmetric form. Let

$$h = c + 2 - s, \quad h' = 2s - 3cs - s^2 + cs^2 - 2c^2 + c.$$

The formulas (2.2) and the identity $g_B = c + \zeta - sx_0$ of Lemma A.4 give

$$\begin{aligned} \rho = 1 - ct, \quad s\eta + c^2t = 2c - 1, \quad cx_0 + st = h, \quad \zeta - sx_0 + c = g_B, \\ c\eta + s\zeta - (1 + cs)t = 2 - 3c - s. \end{aligned} \quad (3.1)$$

The last identity follows by multiplying the formula for ζ by s and substituting $\rho = 1 - ct$. Eliminating η and ζ from this identity gives

$$B \begin{pmatrix} x_0 \\ t \end{pmatrix} = \begin{pmatrix} h \\ h' - s^2g_B \end{pmatrix}, \quad B = \begin{pmatrix} c & s \\ s^3 & -(c+s) \end{pmatrix}. \quad (3.2)$$

Define the projected abscissa $X = cx_0$, the rescaled height $Y = t/c$, and

$$z = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad K = \text{diag}(1/c, c), \quad w = cs, \quad M_w = I + wQ, \quad D_w = \det M_w = 1 + w - w^2, \quad (3.3)$$

so that $(x_0, t)^\top = Kz$. The denominator D of T_B in (2.4) is D_w .

Lemma 3.1 (Symmetric normal form). *Exactly one matrix $J_a = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}$ makes J_aBK symmetric: it has $a = s/c$ and satisfies*

$$J_{s/c}BK = M_w. \quad (3.4)$$

With

$$\xi = \begin{pmatrix} h \\ (s/c)h - h' \end{pmatrix}, \quad e = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad L = (0, c(c+s)), \quad d = 2 - c, \quad \chi = (0, c), \quad (3.5)$$

every member of the contact family satisfies

$$M_w z = \xi + \lambda e, \quad g_A = Lz - d, \quad g_B = \frac{\lambda}{s^2}, \quad T = 3 + \chi z. \quad (3.6)$$

Proof. The product BK is $\begin{pmatrix} 1 & cs \\ s^3/c & -c(c+s) \end{pmatrix}$. The off-diagonal entries of J_aBK are therefore cs and $a - s^3/c$. Symmetry forces $a = cs + s^3/c = s/c$. The lower-right entry then becomes $s^2 + c^2 + cs = 1 + cs$, proving (3.4). Both changes are invertible: $J_{s/c}^2 = I$ and $c > 0$. Apply $J_{s/c}$ to (3.2) and let $\lambda = s^2g_B$. This gives the first identity of (3.6); the remaining identities follow from $g_A = (c+s)t - (2-c)$ and $t = cY$. $\square$

Uniqueness holds for the fixed coordinates $X = cx_0$, $Y = t/c$ and row operations J_a . Unrestricted invertible changes can transform any invertible 2×2 matrix into $I + w'Q$ for $0 < w' < \varphi$. Here the geometric parameter $w = \sin \theta \cos \theta$ determines the cell shape, with both clearances and the enclosing side retained in (3.6).

3.2. One cell tiles the plane and eleven tile the torus. For $0 < w < 1$ let

$$S = [0, 1]^2, \quad H_w = [w, 1] \times [1, 1+w], \quad F_w = S \cup H_w,$$

a unit square with a rectangle of sides $1 - w$ and w attached to its top edge.

Theorem 3.2 (Square-rectangle lattice tiling). *The translates of F_w by $M_w\mathbb{Z}^2$ cover the plane and have pairwise disjoint interiors. The unit squares among them have density $1/D_w$.*

Proof. Write $a = (1, w)$ and $b = (w, 1 + w)$ for the columns of M_w , so that a lattice vector has the form

$$(x, y) = M_w(m, n) = (m + wn, wm + (1 + w)n).$$

Let $(m, n) \neq (0, 0)$. If m and n have the same weak sign, one of $|x|$ and $|y|$ is at least one. Otherwise, reverse both signs if necessary to obtain $m \geq 1$ and $n = -j$ with $j \geq 1$. If $m \geq j + 1$,

then $x \geq 1 + (1 - w)j \geq 1$; if $m \leq j$, then $y \leq -j \leq -1$. Distinct translates of S therefore have disjoint interiors. The rectangle H_w lies in the translated unit square $[w, w + 1] \times [1, 2]$, so its lattice translates also have disjoint interiors.

An intersection of $\text{int } S$ with $\text{int}(H_w + v)$ requires

$$-1 < v_x < 1 - w, \quad -1 - w < v_y < 0.$$

In the lattice basis a and $b - a = (w - 1, 1)$ write $v = ra + j(b - a)$, so that

$$r = \frac{v_x + (1 - w)v_y}{D_w}, \quad j = \frac{v_y - wv_x}{D_w}.$$

Since $0 < w < 1$, these inequalities give $-2 < r < 1$ and $-2 < j < 1$. Thus $r, j \in \{-1, 0\}$ and $v \in \{0, -a, -(b - a), -b\}$. These vectors satisfy, respectively, $v_y = 0$, $v_x = -1$, $v_x = 1 - w$, and $v_y = -1 - w$. Each contradicts a required strict inequality.

The cell area $1 + w(1 - w) = D_w$ equals the lattice covolume. Since the interiors are disjoint, integration over a fundamental parallelogram shows that the uncovered set has measure zero. Local finiteness makes the union of the closed cells closed, so its complement is open and therefore empty. The square density is $1/D_w$. $\square$

Define

$$C_- = 4I - Q = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}, \quad C_+ = 3I + Q = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}. \quad (3.7)$$

The relation $Q^2 = Q + I$ gives

$$Q^5 - I = Q^3 C_-, \quad Q^5 + I = Q^2 C_+, \quad C_- C_+ = 11I. \quad (3.8)$$

Under $(n_1, n_2) \mapsto n_1 + n_2 \varphi$, identify $\mathbb{Z}^2$ with $\mathcal{O} = \mathbb{Z}[\varphi]$ and Q with multiplication by φ . Unimodularity of Q^3 gives $(Q^5 - I)\mathbb{Z}^2 = C_- Q^3 \mathbb{Z}^2 = C_- \mathbb{Z}^2$, so $C_- \mathbb{Z}^2$ represents the ideal $(\varphi^5 - 1)$ of the fifth Fibonacci return.

Corollary 3.3 (The eleven-cell quotient). *The flat torus*

$$\mathbb{T}_w^+ = \mathbb{R}^2 / (M_w C_- \mathbb{Z}^2)$$

has area $11D_w$ and is tiled by eleven unit squares and eleven rectangles congruent to H_w . Its cells are indexed by

$$\mathbb{Z}^2 / C_- \mathbb{Z}^2 \longrightarrow \mathbb{F}_{11}, \quad (n_1, n_2) \longmapsto n_1 + 4n_2, \quad (3.9)$$

the action $n \mapsto Qn$ on these labels is multiplication by 4, and its orbits are

$$\{0\}, \quad (1, 4, 5, 9, 3), \quad (2, 8, 10, 7, 6).$$

Proof. Since $\det C_- = 11$, the period lattice has index eleven in the cell lattice. The tiling of Theorem 3.2 therefore descends with eleven cells. The map (3.9) is surjective and annihilates both columns of C_- . Its kernel and $C_- \mathbb{Z}^2$ have the same index, so they coincide. For $n = (n_1, n_2)$,

$$(Qn)_1 + 4(Qn)_2 = n_2 + 4(n_1 + n_2) = 4(n_1 + 4n_2) - 11n_2 \equiv 4(n_1 + 4n_2) \pmod{11},$$

and repeated multiplication by 4 gives the displayed orbits. $\square$

These residue labels belong to lattice sites; the planar squares retain the labels $K_0, \dots, K_{10}$.

3.3. One marked point recovers every square. The map $u \mapsto w = cs$ is strictly increasing on $\mathcal{I}$. Its derivative and inverse are

$$w'(u) = \frac{2(u^4 - 6u^2 + 1)}{(1 + u^2)^3} > 0, \quad c = \sqrt{\frac{1 + \sqrt{1 - 4w^2}}{2}}, \quad s = \frac{w}{c}, \quad u = \frac{s}{1 + c}. \quad (3.10)$$

The derivative is positive because $1 - 6u_1^2 > 0$. The inverse uses $c > s$ on $\mathcal{I}$. Let

$$W = w(\mathcal{I}) = \left[\frac{61200}{124609}, \frac{63869400}{129254161} \right] \subset \left(0, \frac{1}{2} \right).$$

Equip $\mathbb{T}_w^+$ with its cell decomposition, a fixed oriented orthonormal frame and the residue labels of Corollary 3.3. Choose the lower left corner of the zero cell's unit square as origin. The

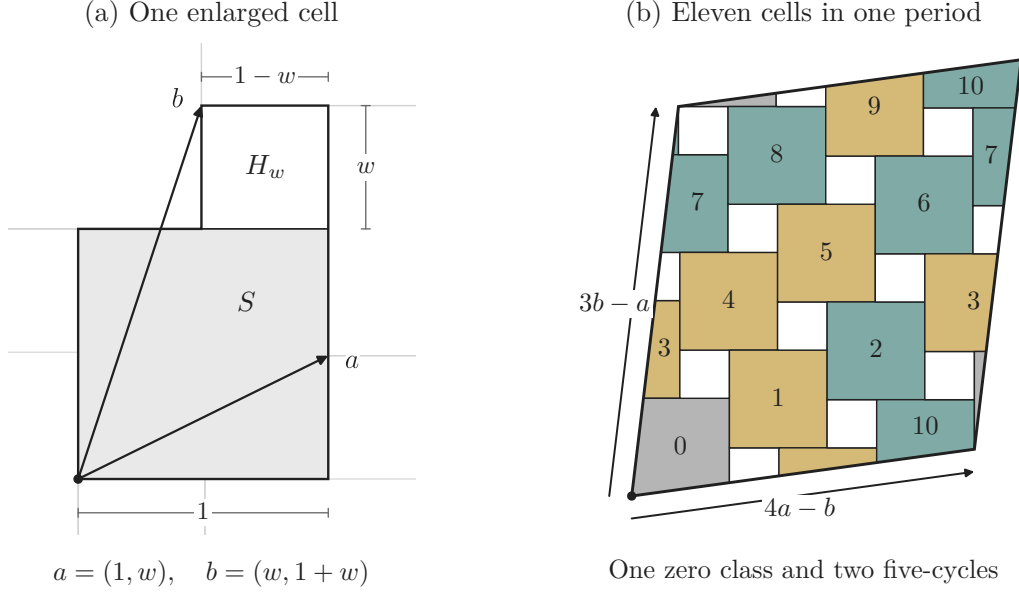


FIGURE 3. The cell and eleven-cell quotient at $w = \sin \theta_* \cos \theta_*$. The unit square S is light gray, the $(1 - w) \times w$ rectangle H_w is white, and neighboring cells have light gray outlines. The translation vectors are $a = (1, w)$ and $b = (w, 1 + w)$. The periods $4a - b$ and $3b - a$ enclose eleven cells. Lattice origins are shown as black dots, and repeated boundary labels identify fragments of one cell. The residue classes $H = \{1, 3, 4, 5, 9\}$ and $-H$ are gold and teal, respectively, and zero is dark gray. Each nonzero class forms a five-cycle under multiplication by four modulo eleven.

shorter side of H_w determines w . Equation (3.10) then determines every coefficient in (3.5). Let $\pi_w : \mathbb{R}^2 \rightarrow \mathbb{T}_w^+$ be the quotient map and define

$$\mathcal{Z}_w = \left\{ \begin{pmatrix} h - s(T - 3) \\ (T - 3)/c \end{pmatrix} : T \in \mathcal{J} \right\}. \quad (3.11)$$

The first equation of (3.6) and $T \in \mathcal{J}$ require $z \in \mathcal{Z}_w$.

Proposition 3.4 (One marked point suffices). *For every $w \in W$, the segment $\mathcal{Z}_w$ has diameter less than $1/40$ and every nonzero period of $\mathbb{T}_w^+$ has length greater than one. The projection π_w is injective on $\mathcal{Z}_w$.*

Proof. On $\mathcal{I}$, the inequality $u^2 < 1/7$ gives $c > 3/4$. Since $s < 1$ and $\mathcal{J}$ has length $3/200$,

$$\text{diam } \mathcal{Z}_w = \frac{3}{200} \sqrt{s^2 + c^{-2}} < \frac{3}{200} \sqrt{1 + \frac{16}{9}} = \frac{1}{40}.$$

The symmetric commuting matrices M_w and C_- are positive definite. Their least eigenvalues satisfy

$$\lambda_{\min}(M_w) = 1 - w/\varphi > \frac{1}{2}, \quad \lambda_{\min}(C_-) = 4 - \varphi > 2.$$

Consequently $\|M_w C_- n\| > \|n\| \geq 1$ for every $n \in \mathbb{Z}^2 \setminus \{0\}$. Two distinct points of $\mathcal{Z}_w$ cannot differ by such a period, proving injectivity. $\square$

A *boundary mark* is a point $p \in \pi_w(\mathcal{Z}_w)$. Its *canonical lift* is the unique $z \in \mathcal{Z}_w$ with $\pi_w(z) = p$. Let $\mathcal{T}^+$ be the family of marked tori with $w \in W$ whose canonical lift satisfies

$$M_w z = \xi + \lambda e, \quad \lambda \geq 0, \quad Lz \geq d, \quad \tau = 3 + \chi z \in \mathcal{J}. \quad (3.12)$$

Theorem 3.5 (Exact recovery from the marked torus). *The map*

$$\mathcal{E} : \mathcal{P} \longrightarrow \mathcal{T}^+, \quad (u, T) \longmapsto \left(\mathbb{T}_{cs}^+, \pi_{cs} \left(\frac{cx_0}{(T-3)/c} \right) \right) \quad (3.13)$$

is a semialgebraic bijection in the coordinates (u, T) and the canonical-lift coordinates (w, z) . It preserves both clearances and the enclosing side:

$$g_A = Lz - d, \quad g_B = \frac{\lambda}{s^2}, \quad T = \tau.$$

Its inverse recovers c , s and u by (3.10) and then

$$\begin{aligned} t &= cY, & T &= 3 + cY, & x_0 &= \frac{X}{c}, \\ \rho &= 1 - ct, & \eta &= \frac{2c - 1 - c^2t}{s}, & \zeta &= sx_0 - c + \frac{\lambda}{s^2}. \end{aligned} \quad (3.14)$$

Proof. Equations (3.6) and Lemma 2.2 show that $\mathcal{E}$ maps $\mathcal{P}$ into $\mathcal{T}^+$ and preserves the three quantities. Conversely, let a marked torus have canonical lift z satisfying (3.12). Recover $u \in \mathcal{I}$ by (3.10) and let $T = 3 + cY \in \mathcal{J}$. The first row is $X + wY = h$, or $cx_0 + st = h$. Thus x_0 , ρ , and η in (3.14) agree with (2.2) at (u, T) .

Let ζ_0 be the value prescribed by (2.2). Apply Lemma 3.1 to obtain $M_w z - \xi = s^2(c + \zeta_0 - sx_0)e$. Comparison with (3.12) gives $\zeta_0 = sx_0 - c + \lambda/s^2$. The recovered pair has $g_A = Lz - d \geq 0$ and $g_B = \lambda/s^2 \geq 0$, so Lemma 2.2 places it in $\mathcal{P}$. Both compositions recover all parameters and the canonical lift, hence the mark by Proposition 3.4. The rational operations and positive square roots prove semialgebraicity. $\square$

Corollary 3.6 (One real parameter remains at fixed cell shape). *For fixed $w \in W$ the canonical lifts of admissible marks form a nondegenerate affine segment,*

$$T \in \left[\max\{T_A(u), T_B(u)\}, \frac{777}{200} \right], \quad z = \begin{pmatrix} h - s(T-3) \\ (T-3)/c \end{pmatrix},$$

so the enclosing side is a function of the marked torus and not of the underlying torus. The minimum of τ over $\mathcal{T}^+$ is T_ , attained at exactly one marked torus.*

Proof. The equations $X = h - wY$ and $T = 3 + cY$, together with Theorem 3.5 and (2.6), give the segment. Its lower endpoint lies below 777/200 by the bounds in the proof of Theorem 2.4. Distinct values of T give distinct lifts, hence distinct marks by Proposition 3.4. The bijection transfers the unique minimum of Theorem 2.4 to $\mathcal{T}^+$. $\square$

3.4. A rectangular chart preserves the Euclidean cells. Let

$$\ell_w = \sqrt{17 - 14w + 10w^2}, \quad q_w = \frac{11D_w}{\ell_w}, \quad \delta_w = \frac{-7 + 20w + 3w^2}{\ell_w}.$$

Proposition 3.7 (The shifted seam of Figure 1). *The rotation*

$$R_w = \frac{1}{\ell_w} \begin{pmatrix} 4 - w & 3w - 1 \\ 1 - 3w & 4 - w \end{pmatrix}$$

represents $\mathbb{T}_w^+$ by the rectangle $[0, \ell_w] \times [0, q_w]$ with the identifications

$$(0, y) \sim (\ell_w, y), \quad (x, 0) \sim ((x + \delta_w) \bmod \ell_w, q_w).$$

The rotation preserves Euclidean cell shapes and orientation and sends the canonical lift of the mark to $R_w z$.

Proof. Since $(4 - w)^2 + (3w - 1)^2 = \ell_w^2$, we have $R_w^T R_w = I$ and $\det R_w = 1$. Also,

$$R_w M_w C_- = \begin{pmatrix} \ell_w & \delta_w \\ 0 & q_w \end{pmatrix}.$$

Subtract a multiple of (δ_w, q_w) to place the vertical coordinate in $[0, q_w]$, then a multiple of $(\ell_w, 0)$ to place the horizontal coordinate in $[0, \ell_w]$. These choices are unique in that order,

proving the seam description. The rotation preserves the oriented cell shapes and sends the canonical lift to $R_w z$. $\square$

3.5. Rational parameters yield square tori. The family $\mathbb{T}_w^+$ extends to every $0 < w < 1$.

Proposition 3.8 (Rational cell shapes). *Let p and q be coprime integers with $0 < p < q$, and let $w = p/q$, $C_{p,q} = (p+q)I - pQ$ and $N_{p,q} = q^2 + pq - p^2$. Then*

$$M_w C_{p,q} = \frac{N_{p,q}}{q} I, \quad \det C_{p,q} = N_{p,q},$$

so the square torus $\mathbb{R}^2/((N_{p,q}/q)\mathbb{Z}^2)$ is tiled by $N_{p,q}$ translates of F_w . For $(p, q) = (1, 3)$ the matrix $C_{1,3}$ is C_- , and $\mathbb{T}_{1/3}^+$ is the square torus of side $11/3$.

Proof. Expansion with $Q^2 = I + Q$ cancels the coefficient of Q ,

$$\left(I + \frac{p}{q}Q\right)((p+q)I - pQ) = \left((p+q) - \frac{p^2}{q}\right)I = \frac{N_{p,q}}{q}I,$$

and $\det C_{p,q} = ((p+q) - p\varphi)((p+q) - p\psi) = N_{p,q}$, where $\psi = 1 - \varphi$. Thus $(N_{p,q}/q)\mathbb{Z}^2$ has index $N_{p,q}$ in the cell lattice. The tiling descends with $N_{p,q}$ cells by Theorem 3.2. For $(p, q) = (1, 3)$, the period factor is C_- and the index is eleven. $\square$

The value $w = 1/3$ lies outside the physical window W . Theorem 7.1 places the square torus at the base point of the geodesic on which all the eleven-cell tori lie.

3.6. Periodic continuation increases the packing density.

Corollary 3.9 (Periodic compression of the eleven-square motif). *For every $(u, T) \in \mathcal{P}$, the same eleven squares extend to a periodic packing with periods $a_o = (A_o, 1)$ and $b_o = (A_o - T, T)$, where $A_o = 1 + (c + \zeta + \rho)/s$. Its period area and square density satisfy*

$$\det(a_o, b_o) = A_o(T - 1) + T < T^2, \quad \frac{11}{A_o(T - 1) + T} > \frac{11}{T^2}.$$

Proof. Proposition B.1 gives the periodic packing. By (B.3), $A_o < T$ and $T > 1$, so $T^2 - [A_o(T - 1) + T] = (T - A_o)(T - 1) > 0$. The period contains eleven embedded unit squares, giving the density. $\square$

Figure 6 shows the smaller period cell at the minimum.

4. QUARTER-TURN DUALITY IDENTIFIES CONJUGATE MARKED TORI

4.1. The quarter-turn preserves both clearances and the minimum. Let

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad N_w = (1 + w)I - wQ = \text{adj } M_w, \quad \mathbb{T}_w^- = \mathbb{R}^2/(N_w C_+ \mathbb{Z}^2). \quad (4.1)$$

The quarter-turn Ω satisfies

$$\Omega Q \Omega^{-1} = I - Q = -Q^{-1}, \quad \Omega M_w \Omega^{-1} = N_w, \quad \Omega C_- \Omega^{-1} = C_+. \quad (4.2)$$

Define the conjugate marking data

$$z^- = \Omega z, \quad \xi^- = \Omega \xi, \quad e^- = \Omega e, \quad L^- = L \Omega^{-1}, \quad \chi^- = \chi \Omega^{-1},$$

and use ΩZ_w as the canonical lift segment on $\mathbb{T}_w^-$. By (4.2), Ω carries the positive period lattice to the negative one, so Proposition 3.4 gives injectivity on the rotated segment. Define $\mathcal{T}^-$ using the rotated frame, cells, labels and origin, with $w \in W$ and canonical lifts satisfying

$$N_w z^- = \xi^- + \lambda e^-, \quad \lambda \geq 0, \quad L^- z^- \geq d, \quad \tau^- = 3 + \chi^- z^- \in \mathcal{J}. \quad (4.3)$$

Theorem 4.1 (Marked quarter-turn duality). *The quarter-turn induces an isometry $\mathbb{T}_w^+ \rightarrow \mathbb{T}_w^-$ carrying the cells to the translates of ΩF_w by $N_w \mathbb{Z}^2$. It gives a bijection $\mathcal{T}^+ \rightarrow \mathcal{T}^-$ preserving both clearances and the enclosing side. The linear maps Q^2 and Q^{-2} descend to $\mathbb{T}_w^+$ and $\mathbb{T}_w^-$, respectively, and satisfy*

$$\Omega \circ Q^2 = Q^{-2} \circ \Omega. \quad (4.4)$$

The three minima agree,

$$\min_{\mathcal{P}} T = \min_{\mathcal{T}^+} \tau = \min_{\mathcal{T}^-} \tau^- = T_*,$$

and each minimum is attained at exactly one member.

Proof. Since $\Omega \mathbb{Z}^2 = \mathbb{Z}^2$, the identities (4.2) give

$$\Omega(M_w C_- \mathbb{Z}^2) = N_w C_+ \mathbb{Z}^2, \quad \Omega(F_w + M_w n) = \Omega F_w + N_w \Omega n.$$

The quarter-turn therefore descends to the stated cell-preserving isometry. The marking data satisfy

$$N_w z^- - \xi^- = \Omega(M_w z - \xi), \quad L^- z^- - d = Lz - d, \quad \chi^- z^- = \chi z.$$

These identities preserve both clearances and the enclosing side; Ω^{-1} gives the inverse marked map. Squaring the first identity of (4.2) proves (4.4). The maps $Q^{\pm 2}$ preserve $\mathbb{Z}^2$ and commute with the period matrices, which are polynomials in Q . Thus they descend to the tori. The minimum statement follows from Theorems 2.4 and 3.5. $\square$

4.2. Positive commuting returns in $SL_2(\mathbb{Z})$ are powers of Q^2 .

Lemma 4.2 (The primitive commuting return). *Let $U \in SL_2(\mathbb{Z})$ commute with Q , have positive eigenvalues, and expand in the eigendirection with eigenvalue φ . Then $U = Q^{2k}$ for a unique integer $k \geq 1$. In particular, Q^2 is the least expanding such matrix.*

Proof. Commutation makes both eigendirections of Q invariant under U . Its eigenvalues are $\ell > 1$ in the φ -direction and ℓ^{-1} in the other. Choose $k \geq 0$ with $1 \leq \ell \varphi^{-2k} < \varphi^2$. The matrix $U_0 = Q^{-2k} U$ belongs to $SL_2(\mathbb{Z})$ and has eigenvalues $\ell_0 = \ell \varphi^{-2k}$ and ℓ_0^{-1} . Hence

$$2 \leq \text{tr } U_0 = \ell_0 + \ell_0^{-1} < \varphi^2 + \varphi^{-2} = 3.$$

The integer trace must equal two, forcing $\ell_0 = 1$. Since U_0 is diagonal in the eigendirections of Q , it equals I . Thus $U = Q^{2k}$. Expansion gives $k \geq 1$, and the eigenvalue φ^{2k} determines k uniquely. $\square$

4.3. Piecewise translations realize the cell permutations. Cellwise translations realize the permutations of lattice sites induced by $Q^{\pm 2}$ while preserving cell shapes. To assign boundaries, define

$$\hat{F}_w = [0, 1)^2 \cup ([w, 1) \times [1, 1 + w)).$$

Its lattice translates partition the plane. A boundary point belongs to the cell containing its displacement by $\varepsilon(1, 1)$ for every sufficiently small $\varepsilon > 0$. Let $\varepsilon_1 = (1, 0)^\top$. On the respective tori, define

$$\begin{aligned} S_+(p + M_w n) &= p + M_w(n + \varepsilon_1), & P_+(p + M_w n) &= p + M_w Q^2 n, & p &\in \hat{F}_w, \\ S_-(q + N_w m) &= q + N_w(m + \varepsilon_1), & P_-(q + N_w m) &= q + N_w Q^{-2} m, & q &\in \Omega \hat{F}_w. \end{aligned} \quad (4.5)$$

Proposition 4.3 (Conjugate exchanges of the eleven cells). *The four maps (4.5) are well defined piecewise translations that preserve the square and the rectangle inside every cell. With $S = S_\pm$ and $P = P_\pm$,*

$$S^{11} = P^5 = 1, \quad PSP^{-1} = S^5. \quad (4.6)$$

On $\mathbb{T}_w^-$, the site $N_w m$ has residue label $m_1 + 8m_2$. The quarter-turn sends positive label k to negative label $8k$ and satisfies

$$\Omega S_+ \Omega^{-1} = S_-^8, \quad \Omega P_+ \Omega^{-1} = P_-.$$

Proof. Changing n by C_-v changes M_wQ^2n by the period $M_wC_-Q^2v$. Similarly, changing m by C_+v changes $N_wQ^{-2}m$ by the period $N_wC_+Q^{-2}v$. The translation maps are also independent of representatives. The half-open convention therefore makes all four maps well defined, each a translation on every cell.

On positive labels, S_+ adds one and P_+ multiplies by $4^2 = 5$, by Corollary 3.3. The map $m \mapsto m_1 + 8m_2$ is surjective and annihilates the columns of C_+ , so it labels the negative torus. Since $Q^{-2} = 2I - Q$, the label of $Q^{-2}m$ is

$$2m_1 - m_2 + 8(m_2 - m_1) = -6m_1 + 7m_2 \equiv 5(m_1 + 8m_2) \pmod{11}.$$

Thus S_- also adds one and P_- multiplies by five. These permutations satisfy (4.6), because five has order five modulo eleven and $5(k+1) = 5k + 5$. The cell maps satisfy the same relations because they preserve internal coordinates.

The site $\Omega n = (-n_2, n_1)$ has negative label $-n_2 + 8n_1 \equiv 8(n_1 + 4n_2)$. Moreover, $\Omega \varepsilon_1 = (0, 1)^T$ has negative label eight, proving the first conjugation. The second follows from (4.4). $\square$

Admissible marks must still satisfy (3.12) and (4.3). The group generated by these exchanges is identified in Corollary 6.5.

4.4. Every affine return equation recovers the same packing. For fixed $w \in W$, let

$$A = -wQ, \quad F_\lambda(z) = Az + \xi + \lambda e, \quad S_n = \sum_{j=0}^{n-1} A^j.$$

The eigenvalues φ and $-\varphi^{-1}$ of Q give $\|A\|_2 = \varphi w < 1$, and $w > 0$ makes A invertible.

Proposition 4.4 (Equivalence of affine return equations). *For every integer $n \geq 1$, the equations*

$$(I - A)z = \xi + \lambda e, \tag{4.7}$$

$$(I - A^n)z = S_n(\xi + \lambda e), \tag{4.8}$$

$$(I - A^{-1})z = -A^{-1}(\xi + \lambda e) \tag{4.9}$$

have exactly the same pairs (z, λ) . With

$$\lambda \geq 0, \quad Lz \geq d, \quad \tau = 3 + \chi z \in \mathcal{J}$$

retained, each equation reconstructs the same members of $\mathcal{P}$, with

$$g_A = Lz - d, \quad g_B = \lambda/s^2, \quad T = \tau.$$

In particular, the cubic return equation is

$$(I + w^3Q^3)z = (I - wQ + w^2Q^2)(\xi + \lambda e). \tag{4.10}$$

Proof. These are the fixed-point equations of F_λ , F_λ^n and F_λ^{-1} , respectively. The identities

$$I - A^n = S_n(I - A), \quad I - A^{-1} = -A^{-1}(I - A)$$

prove equivalence: $\|A\|_2 < 1$ makes $I - A$ and $I - A^n$ invertible, and hence S_n invertible. The operations retain z and λ , so Theorem 3.5 recovers the same packing, clearances and enclosing side. Finally, $A^3 = -w^3Q^3$ and $S_3 = I - wQ + w^2Q^2$ give (4.10). $\square$

Corollary 4.5 (Fibonacci expansion and convergence). *For each $\lambda \in \mathbb{R}$, the unique fixed point of F_λ is*

$$z_\lambda = \sum_{j=0}^{\infty} (-w)^j \begin{pmatrix} F_{j-1} & F_j \\ F_j & F_{j+1} \end{pmatrix} (\xi + \lambda e), \quad F_{-1} = 1, \quad F_0 = 0, \quad F_{j+1} = F_j + F_{j-1}.$$

For every $z \in \mathbb{R}^2$ and integer $n \geq 0$,

$$\|F_\lambda^n(z) - z_\lambda\|_2 \leq (\varphi w)^n \|z - z_\lambda\|_2.$$

Proof. Induction identifies the displayed Fibonacci matrix with Q^j . Since $\|A\|_2 = \varphi w < 1$, the Neumann series for $(I - A)^{-1}$ gives the unique fixed point. The identity $F_\lambda^n(z) - z_\lambda = A^n(z - z_\lambda)$ proves the bound. $\square$

The quarter-turn conjugate has linear part $\Omega A \Omega^{-1} = wQ^{-1}$, whereas inverse iteration has linear part $A^{-1} = -w^{-1}Q^{-1}$.

5. THE CUBIC RETURN SELECTS AN ELEVEN-ELEMENT QUOTIENT

Let $\psi = 1 - \varphi = -\varphi^{-1}$ be the conjugate of φ . Then $N(m + n\varphi) = m^2 + mn - n^2$. The Lucas numbers $L_n = \varphi^n + \psi^n$ satisfy $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$. Define the n th return quotient

$$R_n = \mathcal{O}/(\varphi^n - 1), \quad n \geq 1,$$

and write φ also for its residue. Since $\varphi - 1 = \varphi^{-1}$ and $\varphi^2 - 1 = \varphi$ are units, R_1 and R_2 are zero.

Lemma 5.1 (The order of a return quotient). *For $n \geq 3$,*

$$|R_n| = L_n - 1 - (-1)^n. \quad (5.1)$$

The first eight values appear in Table 2.

Proof. Multiplication by $\varphi^n - 1$ on the lattice $\mathcal{O}$ has determinant

$$N(\varphi^n - 1) = (\varphi\psi)^n - L_n + 1 = (-1)^n - L_n + 1.$$

Its absolute value is the index of the ideal. Since $L_n \geq 4$ for $n \geq 3$, this index is $L_n - 1 - (-1)^n$. $\square$

n	3	4	5	6	7	8	9	10
L_n	4	7	11	18	29	47	76	123
$ R_n $	4	5	11	16	29	45	76	121

TABLE 2. Orders of the first eight nonzero return quotients $R_n = \mathcal{O}/(\varphi^n - 1)$.

5.1. The fifth return quotient recovers the eleven cell labels. Define

$$\mathfrak{p}_- = (4 - \varphi), \quad \mathfrak{p}_+ = (3 + \varphi).$$

Proposition 5.2 (The two prime quotients). *The ideals $\mathfrak{p}_-$ and $\mathfrak{p}_+$ are distinct conjugate prime ideals of norm eleven. They satisfy*

$$\mathfrak{p}_- = (\varphi^5 - 1), \quad \mathfrak{p}_+ = (\varphi^5 + 1), \quad \mathfrak{p}_- \mathfrak{p}_+ = (11). \quad (5.2)$$

The maps

$$\mathcal{O}/\mathfrak{p}_- \longrightarrow \mathbb{F}_{11}, \quad [m + n\varphi] \longmapsto m + 4n, \quad \mathcal{O}/\mathfrak{p}_+ \longrightarrow \mathbb{F}_{11}, \quad [m + n\varphi] \longmapsto m + 8n$$

are ring isomorphisms. Thus $R_5 \cong \mathbb{F}_{11}$, and the label map (3.9) of the eleven-cell torus is reduction modulo $\mathfrak{p}_-$.

Proof. The relation $\varphi^2 = 1 + \varphi$ gives $\varphi^3 = 1 + 2\varphi$ and $\varphi^5 = 3 + 5\varphi$, so

$$\varphi^5 - 1 = \varphi^3(4 - \varphi), \quad \varphi^5 + 1 = \varphi^2(3 + \varphi), \quad (4 - \varphi)(3 + \varphi) = 11.$$

Since φ is a unit, these identities prove (5.2). Both generators have norm eleven. The residues 4 and 8 are the two roots of $X^2 - X - 1$ modulo eleven. Evaluation at either root is a surjective ring homomorphism to $\mathbb{F}_{11}$ that annihilates the corresponding generator. The index calculation identifies its kernel with the stated ideal. The quotients are fields, so the ideals are prime. Conjugation sends $4 - \varphi$ to $4 - \psi = 3 + \varphi$. The coefficient identification $\mathbb{Z}^2 = \mathcal{O}$ in Section 3 gives the assertion about cell labels. $\square$

5.2. The tenth return combines the conjugate quotients.

Proposition 5.3 (The tenth return module). *The tenth return quotient is $V = R_{10} = \mathcal{O}/(11)$. The map*

$$V \xrightarrow{\sim} \mathbb{F}_{11} \times \mathbb{F}_{11}, \quad [m + n\varphi] \mapsto (\alpha, \beta) = (m + 4n, m + 8n), \quad (5.3)$$

is a ring isomorphism, with inverse coefficients $m = 2\alpha - \beta$ and $n = 3(\beta - \alpha)$ modulo eleven. In these coordinates,

$$Q(\alpha, \beta) = (4\alpha, 8\beta), \quad \Omega(\alpha, \beta) = (4\beta, 8\alpha), \quad Q^5(\alpha, \beta) = (\alpha, -\beta), \quad (5.4)$$

and the two quotient lattices satisfy

$$C_- \mathbb{Z}^2 \cap C_+ \mathbb{Z}^2 = 11\mathbb{Z}^2. \quad (5.5)$$

Proof. The identity $\varphi^{10} - 1 = 11\varphi^5$ gives

$$Q^{10} - I = 11Q^5, \quad (5.6)$$

so $(\varphi^{10} - 1) = (11)$ and $V = \mathcal{O}/(11)$. The distinct maximal ideals of Proposition 5.2 are comaximal. Their intersection is their product (11) , proving (5.5); the Chinese remainder theorem gives (5.3). Substitution verifies the inverse coefficients. Multiplication by φ acts by 4 and 8 in the two factors. The formula $\Omega(m, n) = (-n, m)$ gives the second action in (5.4), and $4^5 = 1, 8^5 = -1$ in $\mathbb{F}_{11}$ give the third. $\square$

5.3. The fifth return fixes eleven cells and exchanges fifty-five pairs. For $0 < w < 1$ define

$$\widehat{\mathbb{T}}_w^+ = \mathbb{R}^2 / (11M_w \mathbb{Z}^2), \quad \widehat{\mathbb{T}}_w^- = \mathbb{R}^2 / (11N_w \mathbb{Z}^2).$$

Proposition 5.4 (The fifth return on the 121-cell cover). *Each torus $\widehat{\mathbb{T}}_w^\pm$ is tiled by 121 translates of its square-rectangle cell, indexed by V in the coordinates (5.3). The natural maps $\widehat{\pi}_\pm : \widehat{\mathbb{T}}_w^\pm \rightarrow \mathbb{T}_w^\pm$ are degree-eleven coverings and local isometries. Their cell-label maps are $(\alpha, \beta) \mapsto \alpha$ and $(\alpha, \beta) \mapsto \beta$, respectively, and $\widehat{\pi}_- \Omega = \Omega \widehat{\pi}_+$. The map*

$$\widehat{P}(p + M_w n) = p + M_w Q^5 n, \quad p \in \widehat{F}_w,$$

is a well-defined piecewise translational involution of $\widehat{\mathbb{T}}_w^+$. It fixes the eleven cells with $\beta = 0$, one over each cell of $\mathbb{T}_w^+$. It exchanges the other 110 cells in fifty-five pairs $\{(\alpha, \beta), (\alpha, -\beta)\}$, each lying over the cell α .

For $w \in W$, a marked point with canonical lift z on $\mathbb{T}_w^+$, or z^- on $\mathbb{T}_w^-$, has a distinguished lift to the cover represented by the same vector. These lifts give bijections of the marked families preserving both clearances and the enclosing side. Figure 4 shows one exchanged pair at $w = w_$.*

Proof. By (3.8), the lattice $11\mathbb{Z}^2 = C_- C_+ \mathbb{Z}^2$ lies in both $C_\pm \mathbb{Z}^2$, with

$$[\mathbb{Z}^2 : 11\mathbb{Z}^2] = 121, \quad [C_\pm \mathbb{Z}^2 : 11\mathbb{Z}^2] = 11.$$

These indices count the cells and give the covering degrees. The covering maps descend from the identity on $\mathbb{R}^2$, so they are local isometries. Proposition 5.2 gives their label maps. The identities $\Omega M_w \Omega^{-1} = N_w$ and $\Omega C_- \Omega^{-1} = C_+$ carry the positive lattices to the negative ones and give the commuting isometries.

The half-open convention fixes the cell containing each point. Replacing n by $n + 11v$ changes $Q^5 n$ by $11Q^5 v$, so $\widehat{P}$ is well defined on the quotient. Equation (5.6) gives $\widehat{P}^2 = \text{id}$. By (5.4), its label action is $(\alpha, \beta) \mapsto (\alpha, -\beta)$. It fixes exactly the eleven cells with $\beta = 0$, which project bijectively to the eleven base cells. The remaining labels form fifty-five exchanged pairs with a common first coordinate.

The covering period lattice is contained in the original period lattice. By Proposition 3.4, projection is injective on $\mathcal{Z}_w$ into the positive torus, hence into the cover. The same holds for $\Omega \mathcal{Z}_w$ on the negative side. The unchanged canonical vectors therefore define the distinguished lifts and bijections of the marked families. Their constraints, clearances and enclosing sides depend on the same vector and w . $\square$

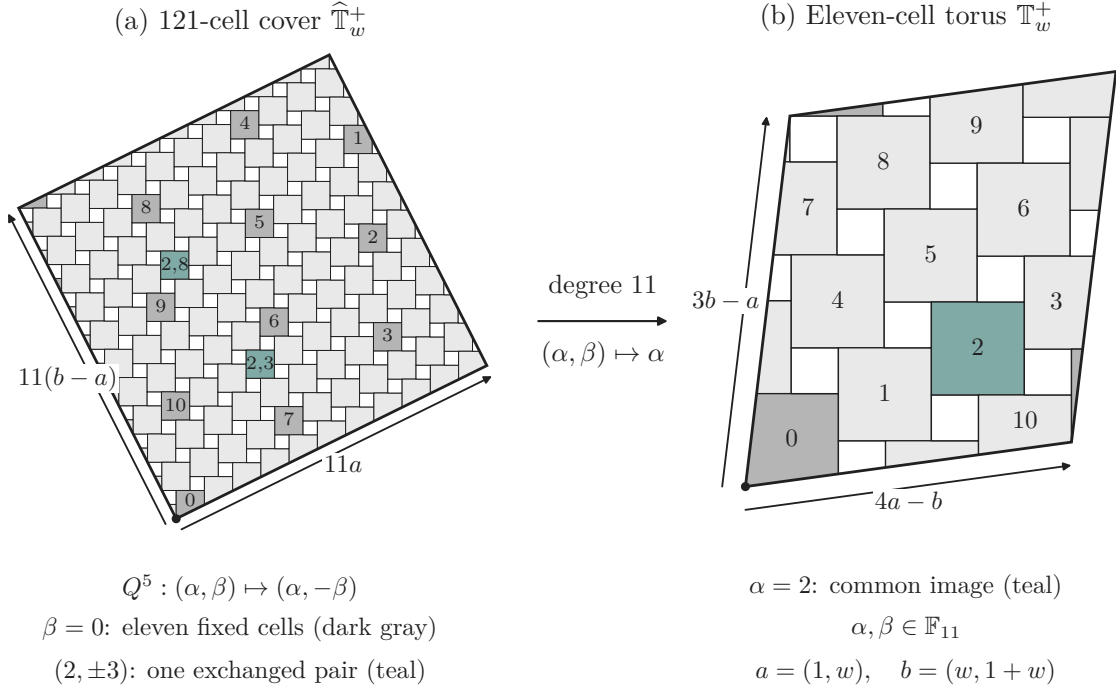


FIGURE 4. The fifth return on the 121-cell cover. The cells of $\widehat{\mathbb{T}}_w^+ = \mathbb{R}^2 / (11M_w \mathbb{Z}^2)$ are indexed by $(\alpha, \beta) \in \mathbb{F}_{11}^2$. The eleven fixed cells have $\beta = 0$ and are shown in dark gray with their α labels. The cells $(2, 3)$ and $(2, 8) = (2, -3)$ form one of the fifty-five exchanged pairs. They and their common image $\alpha = 2 \in -H$ are shown in teal. The base torus $\mathbb{T}_w^+$ is shown at right at the same w and a larger scale, with its zero cell in dark gray. Other unselected cells are light gray. Each origin is marked by a dot, and the period vectors are given beside the boundary arrows. The reciprocal correspondence with pairs of base cells is shown in Figure 5.

5.4. An eleven-element quotient identifies the cubic and inverse primitive returns.

The linear part of the third affine iterate in (4.10) is $-w^3 Q^3$. Define its integral factor

$$R = -Q^3 = \begin{pmatrix} -1 & -2 \\ -2 & -3 \end{pmatrix}, \quad \mu = -\varphi^3 \in \mathcal{O}.$$

Thus the cubic fixed-point equation is

$$(I - w^3 R)z = (I - wQ + w^2 Q^2)(\xi + \lambda e). \quad (5.7)$$

By Proposition 4.4, this equation retains both clearances and the enclosing side. The matrix R acts on the same coefficient lattice as Q and corresponds to multiplication by μ on $\mathcal{O}$.

Theorem 5.5 (The quotient identifying R and Q^{-2}). *The universal abelian quotient of $\mathbb{Z}^2$ on which R and Q^{-2} agree is*

$$\mathbb{Z}^2 / C_+ \mathbb{Z}^2 \cong \mathcal{O} / \mathfrak{p}_+ \cong \mathbb{F}_{11}.$$

Every homomorphism $f : \mathbb{Z}^2 \rightarrow B$ to an abelian group satisfying $fR = fQ^{-2}$ factors uniquely through this quotient. Every nonzero image of such a homomorphism has order eleven. The quarter-turn conjugates this statement to the agreement of Q^{-3} and Q^2 on $\mathbb{Z}^2 / C_- \mathbb{Z}^2$.

Proof. The identities $Q^3 = I + 2Q$ and $Q^{-2} = 2I - Q$ give

$$R - Q^{-2} = -(3I + Q) = -C_+.$$

Hence $fR = fQ^{-2}$ is equivalent to $C_+ \mathbb{Z}^2 \subseteq \ker f$, which is the condition for unique factorization through the quotient. Proposition 5.2 identifies this quotient with $\mathbb{F}_{11}$; its only images are zero

and a group of order eleven. Finally $\Omega^{-1}Q\Omega = -Q^{-1}$ gives

$$\Omega^{-1}R\Omega = Q^{-3}, \quad \Omega^{-1}Q^{-2}\Omega = Q^2, \quad \Omega^{-1}C_+\Omega = C_-. \quad \square$$

The inverse square is distinguished by Lemma 4.2: Q^2 is the least expanding positive determinant-one return commuting with Q . The quotient therefore compares the integral cubic factor in (5.7) with the inverse of that primitive return.

5.5. Cyclic covers recover both eleven-primary modules. For $v \in \mathbb{Z}^2$, define maps of $\mathbb{R}^2 \times \mathbb{R}$ by

$$T_v(x, t) = (x + v, t), \quad h(x, t) = (Rx, t + 1).$$

Since $R \in GL_2(\mathbb{Z})$, these maps satisfy $hT_vh^{-1} = T_{Rv}$. For $n \geq 1$ define

$$\Gamma_n = \langle T_v, h^n : v \in \mathbb{Z}^2 \rangle, \quad X_n = (\mathbb{R}^2 \times \mathbb{R})/\Gamma_n. \quad (5.8)$$

Proposition 5.6 (The torus bundle, its cyclic covers and their homology). *The spaces X_n are closed torus bundles with universal cover $\mathbb{R}^3$ and*

$$\pi_1(X_n) = \Gamma_n \cong \mathbb{Z}^2 \rtimes_{R^n} \mathbb{Z}.$$

For $m \mid n$, inclusion $\Gamma_n \subseteq \Gamma_m$ gives a covering $X_n \rightarrow X_m$ of degree n/m . The manifold X_n is orientable exactly when n is even. Its torsion homology is

$$A_n := \text{Tor } H_1(X_n; \mathbb{Z}) \cong \mathcal{O}/(\mu^n - 1). \quad (5.9)$$

For the double cover $\pi : X_{2n} \rightarrow X_n$, the induced map on torsion is reduction modulo $\mu^n - 1$, and transfer is multiplication by $1 + \mu^n$.

Proof. Every element of Γ_n has the unique form T_vh^{nk} , with $v \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$. Its action is

$$(x, t) \mapsto (R^{nk}x + v, t + nk).$$

Only the identity has a fixed point. A compact set meets only finitely many translates: its bounded time coordinate bounds k , then for each such k its bounded spatial coordinates bound v . The action is therefore free and properly discontinuous. The image of $[0, 1]^2 \times [0, n]$ covers the quotient, proving compactness. Projection to $t \bmod n$ gives the torus bundle with universal cover $\mathbb{R}^3$. The multiplication rule

$$(T_vh^{nk})(T_uh^{n\ell}) = T_{v+R^{nk}u}h^{n(k+\ell)}$$

proves the stated fundamental group. The cosets of Γ_n in Γ_m are represented by $1, h^m, \dots, h^{n-m}$, giving the covering degree. Translation preserves orientation, while h^n has determinant $\det R^n = (-1)^n$, which proves the orientability assertion.

Abelianization identifies v with $R^n v$ and imposes no relation on the stable generator, so

$$H_1(X_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}^2/(R^n - I)\mathbb{Z}^2.$$

The eigenvalues of R are $-\varphi^3$ and $-\psi^3 = \varphi^{-3}$, so neither eigenvalue of R^n equals one. The second summand is finite, and the coefficient identification $\mathbb{Z}^2 = \mathcal{O}$ gives (5.9).

Inclusion preserves fibre translations, so projection is reduction modulo $\mu^n - 1$. The coset representatives $1, h^n$ give the transfer $v \mapsto v + R^n v$, or multiplication by $1 + \mu^n$. This is well defined because $(1 + \mu^n)(\mu^n - 1) = \mu^{2n} - 1$. $\square$

Theorem 5.7 (The first eleven-primary returns). *For a finite abelian group A , write $A[11^\infty]$ for the subgroup of elements killed by a power of eleven. Then $A_n[11^\infty] \neq 0$ if and only if $5 \mid n$. The first such level is five; the first orientable such level is ten. Natural reduction maps restrict to isomorphisms*

$$A_5[11^\infty] \xrightarrow{\sim} \mathcal{O}/\mathfrak{p}_+ \cong \mathbb{F}_{11}, \quad A_{10}[11^\infty] \xrightarrow{\sim} \mathcal{O}/(11) = V. \quad (5.10)$$

In the coordinates (α, β) of (5.3), projection and transfer for $X_{10} \rightarrow X_5$ are

$$\pi_*(\alpha, \beta) = \beta, \quad \pi^!(\beta) = (0, 2\beta). \quad (5.11)$$

Proof. Modulo the two primes over eleven, $\mu = -\varphi^3$ has residues $-4^3 = 2$ and $-8^3 = 5$. Their orders in $\mathbb{F}_{11}^\times$ are ten and five: $2^5 = -1$, $2^2 \neq 1$, and $5^5 = 1$ with $5 \neq 1$. Consequently

$$\det(R^n - I) \equiv (2^n - 1)(5^n - 1) \pmod{11},$$

which vanishes exactly when $5 \mid n$. The order of A_n is $|\det(R^n - I)|$, proving the occurrence statement. By Proposition 5.6, the orientable levels are exactly the even ones.

Since $\mu^5 = -\varphi^{15}$, the group A_5 is $\mathcal{O}/(\varphi^{15} + 1)$. The Lucas recurrence gives $L_{15} = 1364 = 11 \cdot 124$, and therefore

$$|A_5| = |N(\varphi^{15} + 1)| = L_{15} = 11 \cdot 124.$$

The norm identity follows from $(\varphi\psi)^{15} = -1$. Since $\mu^5 \equiv 1 \pmod{\mathfrak{p}_+}$, reduction modulo $\mathfrak{p}_+$ is well defined and surjective. It kills the subgroup of order coprime to eleven and restricts to an isomorphism on the primary subgroup.

For level ten, $\psi^{15} = -\varphi^{-15}$ gives

$$\varphi^{30} - 1 = L_{15}\varphi^{15} = 1364\varphi^{15}.$$

Since φ is a unit, $A_{10} \cong \mathcal{O}/(1364)$. The Chinese remainder theorem gives

$$\mathcal{O}/(1364) \cong \mathcal{O}/(11) \times \mathcal{O}/(124).$$

The second factor has order 124^2 , coprime to eleven. Reduction to the first factor thus restricts to the second isomorphism in (5.10).

Projection followed by reduction modulo $\mathfrak{p}_+$ gives the β -coordinate. In V , $\mu^5 = (2^5, 5^5) = (-1, 1)$, so multiplication by $1 + \mu^5$ reads $(\alpha, \beta) \mapsto (0, 2\beta)$. The transfer formula of Proposition 5.6 proves (5.11). $\square$

The negative flat-torus cover of Proposition 5.4 induces the same projection $(\alpha, \beta) \mapsto \beta$ on the finite modules. The torus-bundle covering has degree two, whereas the flat-torus covering has degree eleven.

6. PAIR REGULARITY SELECTS THE FIFTH RETURN

6.1. Inversion identifies return orbits with cell pairs. On $V = \mathbb{F}_{11} \times \mathbb{F}_{11}$ in the coordinates (5.3), the fifth return and its quarter-turn conjugate act on labels by

$$\Sigma_-(\alpha, \beta) = (\alpha, -\beta), \quad \Sigma_+(\alpha, \beta) = (-\alpha, \beta).$$

Thus $\Sigma_- = Q^5$ and $\Sigma_+ = \Omega Q^5 \Omega^{-1} = -Q^{-5}$. The piecewise exchange $\hat{P}$ of Proposition 5.4 induces Σ_- on cell labels. Define their sets of two-element orbits by

$$V_- = \{(\alpha, \beta) \in V : \beta \neq 0\} / \langle \Sigma_- \rangle, \quad V_+ = \{(\alpha, \beta) \in V : \alpha \neq 0\} / \langle \Sigma_+ \rangle.$$

Let $\binom{\mathbb{F}_{11}}{2}$ denote the unordered pairs of distinct elements of $\mathbb{F}_{11}$. For $a \in \mathbb{F}_{11}^\times$, $b \in \mathbb{F}_{11}$ and $E \in \binom{\mathbb{F}_{11}}{2}$, write $aE + b = \{ax + b : x \in E\}$.

Theorem 6.1 (The reciprocal pair correspondence). *The maps*

$$\Psi_-([\alpha, \beta]) = \{\alpha + \beta^{-1}, \alpha - \beta^{-1}\}, \quad \Psi_+([\alpha, \beta]) = \{\beta + \alpha^{-1}, \beta - \alpha^{-1}\} \quad (6.1)$$

are bijections from V_- and V_+ onto $\binom{\mathbb{F}_{11}}{2}$. With $S(\alpha, \beta) = (\alpha + 1, \beta)$ they satisfy

$$\Psi_-(Qv) = 4\Psi_-(v), \quad \Psi_-(Sv) = \Psi_-(v) + 1, \quad \Psi_+(\Omega v) = 8\Psi_-(v), \quad (6.2)$$

and $|V_-| = |V_+| = |\binom{\mathbb{F}_{11}}{2}| = 55$.

Proof. Replacing β by $-\beta$ exchanges the two distinct entries of Ψ_- , so the map is well defined. Its inverse sends $\{x, y\}$ to the class with $\alpha = (x + y)/2$ and $\beta = 2/(x - y)$: interchanging x, y changes only the sign of β . Exchanging the coordinates proves the assertion for Ψ_+ . Since $Q(\alpha, \beta) = (4\alpha, 8\beta)$ and $8^{-1} = -4$ in $\mathbb{F}_{11}$,

$$\Psi_-(Q[\alpha, \beta]) = \{4\alpha \pm (8\beta)^{-1}\} = \{4\alpha \mp 4\beta^{-1}\} = 4\Psi_-([\alpha, \beta]).$$

Translation of α gives the second identity. The relation $\Omega\Sigma_- = \Sigma_+\Omega$ induces $\Omega : V_- \rightarrow V_+$. Using $\Omega(\alpha, \beta) = (4\beta, 8\alpha)$ and $4^{-1} = -8$ gives

$$\Psi_+(\Omega[\alpha, \beta]) = \{8\alpha \pm (4\beta)^{-1}\} = 8\Psi_-([\alpha, \beta]).$$

Each involution fixes eleven elements of V , leaving fifty-five two-element orbits. $\square$

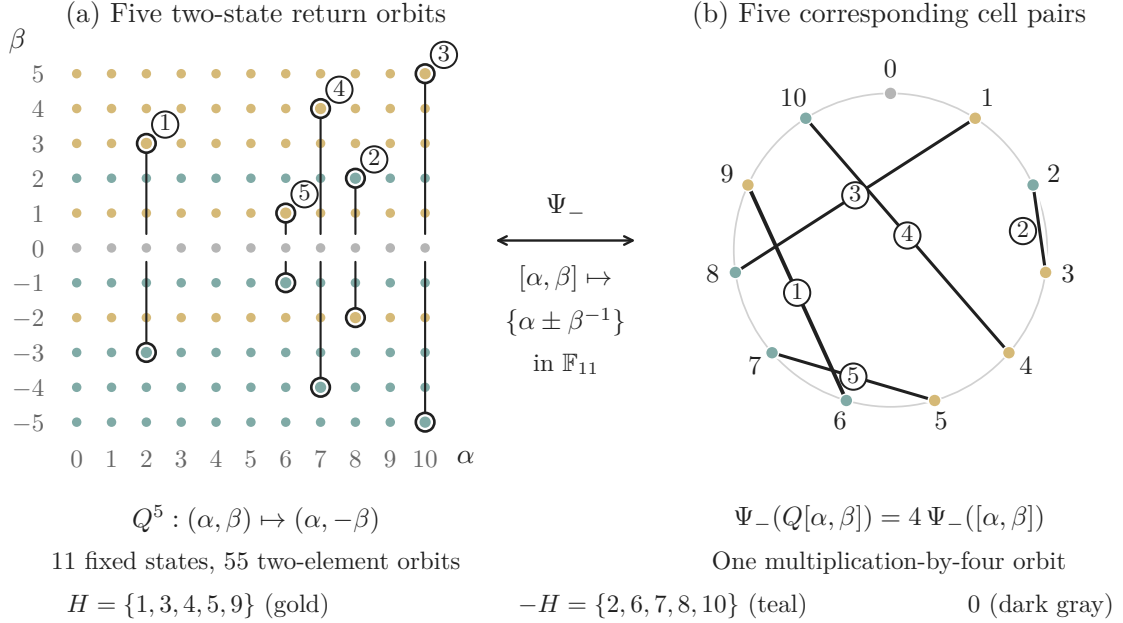


FIGURE 5. A five-cycle of return classes and its image under Ψ_- . Arithmetic is in $\mathbb{F}_{11}$, with $Q(\alpha, \beta) = (4\alpha, 8\beta)$. At left, the two states of each class $[\alpha, \beta] = \{(\alpha, \beta), (\alpha, -\beta)\}$, $\beta \neq 0$, are joined by a numbered vertical segment. At right, its image pair of base-cell labels is joined by an edge with the same number. The classes H , $-H$ and zero are shown in gold, teal and dark gray, determined by β at left and by the cell label at right. All five classes and image pairs appear in Table 3.

The inversion in (6.1) turns the multiplier 8 into -4 . Changing the sign of the half-difference exchanges the two endpoints, so Q multiplies the unordered pair by 4. The map Q acts on labels and return orbits. The piecewise exchanges act on the cells themselves.

Proposition 6.2 (The eleven five-cycles). *Let $H = \langle 4 \rangle = \{1, 3, 4, 5, 9\} \subset \mathbb{F}_{11}^\times$. Every pair has a unique expression $\{\alpha - \mu, \alpha + \mu\}$ with $\alpha \in \mathbb{F}_{11}$ and $\mu \in H$. Under Ψ_- one has $\mu = \beta^4$. The action of Q on V_- has eleven orbits of length five, whose images are*

$$\{\{4^j(a-1), 4^j(a+1)\} : 0 \leq j < 5\}, \quad a \in \mathbb{F}_{11}. \quad (6.3)$$

Proof. The nonzero squares form H . Its odd order excludes -1 , so H and $-H$ partition $\mathbb{F}_{11}^\times$. Every pair has a unique midpoint and exactly one half-difference in H . Fermat's identity gives $\beta^5 = \pm 1$, hence $\{\pm\beta^{-1}\} = \{\pm\beta^4\}$ with $\beta^4 \in H$. In the coordinates (α, μ) , Q acts by $(\alpha, \mu) \mapsto (4\alpha, 4\mu)$. Since $\mu \neq 0$ and 4 has order five, every orbit has length five. Scaling by $\mu^{-1} \in H$ gives its unique member with half-difference one and midpoint $a = \alpha/\mu$, proving (6.3). $\square$

Corollary 6.3 (Eleven near-perfect matchings). *For each $\alpha \in \mathbb{F}_{11}$, the five two-element return orbits over the cell label α map under Ψ_- to*

$$\mathcal{M}_\alpha = \{\{\alpha - \mu, \alpha + \mu\} : \mu \in H\}. \quad (6.4)$$

Each $\mathcal{M}_\alpha$ is a near-perfect matching: its five pairs cover every label except α exactly once. The eleven matchings partition all fifty-five edges of the complete graph on $\mathbb{F}_{11}$.

Proof. At fixed α , the ten nonzero values of β form five orbits under $\beta \mapsto -\beta$. Their images are (6.4), by Proposition 6.2. Since H and $-H$ partition $\mathbb{F}_{11}^\times$, the ten endpoints $\alpha \pm \mu$ are distinct and exhaust $\mathbb{F}_{11} \setminus \{\alpha\}$. Each pair $\{x, y\}$ belongs to the unique matching with midpoint $\alpha = (x + y)/2$. $\square$

The matchings group pairs by midpoint, and the five-cycles group them by iteration of Q . Multiplication by four sends $\mathcal{M}_\alpha$ to $\mathcal{M}_{4\alpha}$.

Number in Figure 5	1	2	3	4	5
Class $[\alpha, \beta]$	$[2, 3]$	$[8, 2]$	$[10, 5]$	$[7, 4]$	$[6, 1]$
Image under Ψ_-	$\{6, 9\}$	$\{2, 3\}$	$\{1, 8\}$	$\{4, 10\}$	$\{5, 7\}$

TABLE 3. The orbit in Figure 5, with $[\alpha, \beta] = \{(\alpha, \beta), (\alpha, -\beta)\}$ in $\mathbb{F}_{11}^2$. Successive columns follow Q on classes and multiplication by four on image pairs.

6.2. Transitive pair actions occur exactly for $n = 3, 4, 5$. An action is *regular* if it is free and transitive. For $n \geq 3$ let G_n be the subgroup of $\text{Sym}(R_n)$ generated by $x \mapsto x + 1$ and $x \mapsto \varphi x$.

Theorem 6.4 (Classification of pair actions on return quotients). *The action of G_n on $\binom{R_n}{2}$ is transitive if and only if $n \in \{3, 4, 5\}$, and regular if and only if $n = 5$. The three transitive cases have the parameters in Table 4. For $n = 5$, the group is $G_5 = \{x \mapsto ax + b : a \in H, b \in \mathbb{F}_{11}\}$.*

n	R_n	$ G_n $	$ \binom{R_n}{2} $	Pair stabilizer order
3	$\mathbb{F}_4$	12	6	2
4	$\mathbb{F}_5$	20	10	2
5	$\mathbb{F}_{11}$	55	55	1

TABLE 4. The three transitive actions of G_n on unordered pairs of distinct elements of R_n . The action is regular exactly when the pair stabilizer has order one.

Proof. Let $N_n = |R_n|$ and let d_n be the multiplicative order of φ in R_n . Since $\varphi^n = 1$, one has $d_n \mid n$. Conjugation by multiplication by φ sends translation by 1 to translation by φ . Since 1, φ generate the additive group of R_n , all translations occur in G_n . Thus

$$G_n = \{x \mapsto \varphi^j x + b : b \in R_n, 0 \leq j < d_n\}.$$

The values at 0 and 1 determine b and φ^j , so the displayed maps are distinct and $|G_n| = N_n d_n \leq n N_n$.

For $n \geq 6$, $L_n - 2 > 2n + 1$. At $n = 6$ this is $16 > 13$. The increments $L_{n+1} - L_n = L_{n-1} > 2$ preserve the inequality. By Lemma 5.1, $N_n \geq L_n - 2$, so

$$|G_n| \leq n N_n < \frac{N_n(N_n - 1)}{2} = |\binom{R_n}{2}|.$$

No orbit can exceed $|G_n|$, which excludes transitivity for $n \geq 6$.

For $n = 3$, the identity $\varphi^3 - 1 = 2\varphi$ gives $R_3 = \mathcal{O}/(2) = \mathbb{F}_2[X]/(X^2 + X + 1) \cong \mathbb{F}_4$, since $X^2 + X + 1$ has no root in $\mathbb{F}_2$. The residue of φ has order three. For $n = 4$, evaluation at $\varphi = 3$ modulo five annihilates $\varphi^4 - 1 = 3\varphi + 1$ and maps R_4 onto $\mathbb{F}_5$. Since $N_4 = 5$, this is an isomorphism, with multiplier of order four. For $n = 5$, Proposition 5.2 gives $R_5 \cong \mathbb{F}_{11}$ with $\varphi = 4$ of order five.

For $n = 3, 4$, the powers of φ exhaust the field's multiplicative group, so G_n is the full affine group. Exactly one affine map sends an ordered pair (x, y) of distinct points to another such pair (x', y') : its multiplier is $(y' - x')/(y - x)$ and its translation is $x' - (y' - x')x/(y - x)$. The action on unordered pairs is therefore transitive with stabilizer order two.

For $n = 5$, an affine map stabilizing an unordered pair either fixes both endpoints and is the identity, or exchanges them. In the latter case, subtraction gives multiplier -1 , which cannot belong to the odd-order group H . The action is free, and $|G_5| = 55 = \binom{11}{2}$ makes it transitive. $\square$

6.3. Cell exchanges act regularly on the fifty-five pairs.

Corollary 6.5 (The cell-exchange group). *For either sign, the maps $S_\pm$ and $P_\pm$ of Proposition 4.3 generate a group of piecewise translations of $\mathbb{T}_w^\pm$. This group is isomorphic to G_5 and acts regularly on the fifty-five unordered pairs of cells of the torus.*

Proof. By Proposition 4.3, the maps $S^b P^j$, $0 \leq b < 11$, $0 \leq j < 5$, induce the fifty-five distinct permutations $k \mapsto 5^j k + b$. The relations (4.6) show that they exhaust the group. Since $\langle 5 \rangle = \langle 4 \rangle = H$, this label action is G_5 and is regular on pairs by Theorem 6.4. Each map preserves the internal coordinate of every cell, so its label permutation determines it. The group of maps is therefore isomorphic to G_5 . $\square$

The exchanges preserve each cell's square and rectangle but can be discontinuous at cell boundaries and need not preserve admissible marks. The physical squares remain subject to the separation conditions of Section 2.

6.4. The torus-bundle group surjects onto the affine group. Let $a = T_{(1,0)}$, $c = T_{(0,1)}$ and $t = h$ in the torus-bundle group Γ_1 of (5.8). Its presentation is

$$\Gamma_1 = \langle a, c, t : [a, c] = 1, tat^{-1} = a^{-1}c^{-2}, tct^{-1} = a^{-2}c^{-3} \rangle. \quad (6.5)$$

Proposition 6.6 (The fifty-five-element affine quotient). *On $\mathbb{F}_{11}$ let $S(x) = x + 1$ and $P(x) = 5x$. The assignments*

$$a \mapsto S, \quad c \mapsto S^8, \quad t \mapsto P \quad (6.6)$$

define an epimorphism $\Gamma_1 \rightarrow G_5$. Its kernel consists of the elements $T_{(m,n)}h^k$ with

$$m + 8n \equiv 0 \pmod{11}, \quad k \equiv 0 \pmod{5}.$$

The induced action on the fifty-five unordered pairs of distinct elements of $\mathbb{F}_{11}$ is transitive, and every stabilizer equals this kernel. The same quotient acts by cell exchanges on either eleven-cell torus.

Proof. Define $\ell(m, n) = m + 8n \pmod{11}$. Direct multiplication gives

$$\ell(R(m, n)) = -17m - 26n = 5(m + 8n) = 5\ell(m, n) \quad \text{in } \mathbb{F}_{11}.$$

Hence the formula

$$T_v h^k \mapsto S^{\ell(v)} P^k$$

respects the multiplication rule in Proposition 5.6, since $P^k S^b P^{-k} = S^{5^k b}$. This defines a homomorphism with generator images (6.6). It is surjective because S and P generate G_5 , with $\langle 5 \rangle = H$. An affine map $x \mapsto 5^k x + \ell(v)$ is the identity exactly when $5^k = 1$ and $\ell(v) = 0$. The order of 5 is five, proving the kernel description. By Theorem 6.4 the action of G_5 on pairs is regular, so every stabilizer in Γ_1 is the kernel. Finally Corollary 6.5 realizes S, P by the maps $S_\pm, P_\pm$ on the cells. $\square$

6.5. Abelianizing the fibre preserves first homology. Let $F = F(a, c)$ be the free group on two generators.

Lemma 6.7 (A free-group automorphism with cubic abelianization). *The homomorphism $\rho : F \rightarrow F$ defined by*

$$\rho(a) = c^{-1}a^{-1}c^{-1}, \quad \rho(c) = c^{-1}a^{-1}c^{-1}a^{-1}c^{-1}$$

is an automorphism. Its inverse is

$$\rho^{-1}(a) = ac^{-1}ac^{-1}a, \quad \rho^{-1}(c) = a^{-1}ca^{-1},$$

and its action on $F_{\text{ab}} = \mathbb{Z}^2$ is $R = -Q^3$.

Proof. Let $A = \rho(a)$ and $C = \rho(c)$. Define a homomorphism $\sigma : F \rightarrow F$ by $U = \sigma(a) = ac^{-1}ac^{-1}a$ and $V = \sigma(c) = a^{-1}ca^{-1}$. Free reduction gives

$$AC^{-1} = ac, \quad A^{-1}C = a^{-1}c^{-1}, \quad V^{-1}U^{-1} = ca^{-1}.$$

Consequently

$$\begin{aligned} \rho\sigma(a) &= (AC^{-1})^2 A = (ac)^2 A = a, & \rho\sigma(c) &= (A^{-1}C)A^{-1} = c, \\ \sigma\rho(a) &= (V^{-1}U^{-1})V^{-1} = a, & \sigma\rho(c) &= (V^{-1}U^{-1})^2 V^{-1} = c. \end{aligned}$$

Thus $\rho\sigma = \sigma\rho = \text{id}$. The exponent sums of a and c in the two images are the columns $(-1, -2)^\top$ and $(-2, -3)^\top$, which give R . $\square$

For $n \geq 1$ let

$$\Pi_n = F \rtimes_{\rho^n} \langle t^n \rangle = \langle a, c, t^n : t^n x t^{-n} = \rho^n(x), x = a, c \rangle.$$

The words for ρ define a based homotopy equivalence of the two-petalled rose, with inverse given by the words for ρ^{-1} . Its mapping torus has fundamental group Π_1 . The cyclic cover with suspension exponent divisible by n has fundamental group Π_n .

Proposition 6.8 (Abelianizing the fibre preserves first homology). *Abelianization of F induces surjections $\Pi_n \rightarrow \Gamma_n$ and isomorphisms on first homology*

$$H_1(\Pi_n; \mathbb{Z}) \xrightarrow{\sim} H_1(X_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathcal{O}/(\mu^n - 1).$$

These isomorphisms commute with all covering projections and transfers, recovering the eleven-primary groups and maps of Theorem 5.7. The assignments (6.6) define an epimorphism $\Pi_1 \rightarrow G_5$.

Proof. Send a, c to $T_{(1,0)}, T_{(0,1)}$ and t^n to h^n . The relations are preserved because ρ^n abelianizes to R^n , and the images generate Γ_n . Abelianizing the presentation of Π_n imposes exactly the relations $(R^n - I)\mathbb{Z}^2$ on the fibre generators and no relation on t^n . The induced map fixes the generators of the identical presentation for $H_1(X_n; \mathbb{Z})$ in Proposition 5.6, so it is an isomorphism.

For $n = qm$, the inclusions $\Pi_n \subseteq \Pi_m$ and $\Gamma_n \subseteq \Gamma_m$ commute with these maps. The coset representatives $1, t^m, \dots, t^{(q-1)m}$ map to $1, h^m, \dots, h^{(q-1)m}$. On fibre homology, both transfers are therefore multiplication by $\sum_{j=0}^{q-1} R^{mj}$. On the suspension generator, the transfers send $[t^m]$ to $[t^n]$ and $[h^m]$ to $[h^n]$. The projections preserve fibre classes and send $[t^n]$ to $q[t^m]$ and $[h^n]$ to $q[h^m]$. This proves compatibility on all generators. Finally, compose $\Pi_1 \rightarrow \Gamma_1$ with the epimorphism of Proposition 6.6. $\square$

7. THE NORMALIZED TORI LIE ON A SHORTEST CLOSED MODULAR GEODESIC

Let $\mathcal{S}$ be the space of positive definite symmetric real 2×2 matrices of determinant one, with the Riemannian metric

$$ds^2 = \frac{1}{2} \text{tr}(G^{-1} dG G^{-1} dG). \quad (7.1)$$

A lattice of covolume one with oriented basis matrix B has Gram matrix $G = B^\top B \in \mathcal{S}$. The quotient by $G \mapsto U^\top G U$, $U \in SL_2(\mathbb{Z})$, is the modular surface [14].

Let

$$V_0 = \frac{2Q - I}{\sqrt{5}}, \quad C(\sigma) = e^{\sigma V_0}, \quad \sigma(w) = \frac{1}{2} \log \frac{1 + \varphi w}{1 - w/\varphi}.$$

Theorem 7.1 (A shortest closed modular geodesic). *For $0 \leq w \leq 1$,*

$$\frac{M_w}{\sqrt{D_w}} = C(\sigma(w)), \quad \sigma(0) = 0, \quad \sigma(1/2) = \log \varphi, \quad \sigma(1) = 2 \log \varphi. \quad (7.2)$$

The Gram matrices $G(\sigma) = C(\sigma)^2$ project to a shortest closed geodesic of the modular surface, of length $4 \log \varphi = 2 \operatorname{arccosh}(3/2)$. The area-normalized eleven-cell tori lie on the same curve:

$$\frac{M_w C_-}{\sqrt{11 D_w}} = C(\sigma(w) + \kappa), \quad \kappa = \frac{1}{2} \log \frac{4 - \varphi}{3 + \varphi} = -\sigma(1/3), \quad (7.3)$$

so the square torus $\mathbb{T}_{1/3}^+$ of Proposition 3.8 is the base point $\sigma = 0$ of the curve.

Proof. The relation $Q^2 = Q + I$ gives $V_0^\top = V_0$, $V_0^2 = I$, and $\det V_0 = -1$. The orthogonal vectors $(1, \varphi)$ and $(1, -\varphi^{-1})$ are eigenvectors of V_0 with eigenvalues 1 and -1 . In these directions, M_w has eigenvalues $1 + \varphi w$ and $1 - w/\varphi$. They are positive on $[0, 1]$ and have product D_w . Dividing by $\sqrt{D_w}$ gives $e^{\sigma(w)}$ and $e^{-\sigma(w)}$, proving (7.2). The stated parameter values follow from

$$\frac{1 + \varphi/2}{1 - 1/(2\varphi)} = \varphi^2, \quad \frac{1 + \varphi}{1 - 1/\varphi} = \varphi^4,$$

using $1/\varphi = \varphi - 1$.

Write $G = y^{-1} \begin{pmatrix} 1 & x \\ x & x^2 + y^2 \end{pmatrix}$, with $y > 0$. Substitution into (7.1) gives the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. For $G = e^{2\sigma V_0} = \cosh(2\sigma)I + \sinh(2\sigma)V_0$,

$$y = \frac{1}{\cosh 2\sigma - \sinh 2\sigma/\sqrt{5}}, \quad x = \frac{2 \sinh 2\sigma/\sqrt{5}}{\cosh 2\sigma - \sinh 2\sigma/\sqrt{5}}, \quad \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{5}{4}.$$

This semicircle is orthogonal to the real axis, hence is a hyperbolic geodesic. Moreover, $G^{-1}G' = 2V_0$, so

$$ds^2 = \frac{1}{2} \operatorname{tr}(4V_0^2) d\sigma^2 = 4 d\sigma^2.$$

Thus the curve has speed two.

The eigenvalues give $C(2 \log \varphi) = Q^2$. Symmetry and commutation yield

$$G(\sigma + 2 \log \varphi) = (Q^2)^\top G(\sigma) Q^2.$$

The curve therefore closes after length $4 \log \varphi$. Every nonconstant closed modular geodesic lifts to a geodesic translated along itself by a hyperbolic $U \in SL_2(\mathbb{Z})$. A real fractional linear change of coordinates conjugates U to $z \mapsto \ell^2 z$, where $\ell > 1$ is the larger eigenvalue modulus. Every path from z to $\ell^2 z$ has length at least the change in $\log \operatorname{Im} z$, namely $2 \log \ell$, attained along the imaginary axis. Thus the translation length is

$$2 \log \ell = 2 \operatorname{arccosh}(|\operatorname{tr} U|/2).$$

The integer $|\operatorname{tr} U|$ is at least three. Since $\operatorname{tr} Q^2 = 3$, this closed geodesic has least length and is primitive: an iterate would have a shorter closed geodesic as its primitive representative.

Finally, C_- has positive eigenvalues $4 - \varphi$ and $4 - \psi = 3 + \varphi$, with product eleven. Hence $C_-/\sqrt{11} = C(\kappa)$, which proves (7.3). The identity $\kappa = -\sigma(1/3)$ follows from

$$\frac{4 - \varphi}{3 + \varphi} = \frac{1 - 1/(3\varphi)}{1 + \varphi/3}.$$

□

The reflection half-period follows from $C(\log \varphi) = V_0 Q$:

$$G(\sigma + \log \varphi) = Q^\top G(\sigma) Q, \quad \det Q = -1.$$

Thus the quotient by $GL_2(\mathbb{Z})$ closes the curve after half the period. The physical window occupies the short arc $\sigma(W) + \kappa \subset [0.142, 0.145]$, near the square torus. The enclosing side remains the marked functional $\tau = 3 + \chi z$ of (3.12).

8. THE SIDE POLYNOMIAL HAS GALOIS GROUP S_8

Let $F = \mathbb{Q}(T_*)$, with T_* the minimum of Theorem 2.4.

Theorem 8.1 (The octic field of the minimum). *The minimal polynomial of T_* over $\mathbb{Q}$ is*

$$f(X) = X^8 - 20X^7 + 178X^6 - 842X^5 + 1923X^4 - 496X^3 - 6754X^2 + 12420X - 6865. \quad (8.1)$$

It has exactly two real zeros, one negative and the other T_ . The Galois group of f over $\mathbb{Q}$ is S_8 . The extension $F/\mathbb{Q}$ has no proper intermediate field, and T_* is not expressible in radicals over $\mathbb{Q}$.*

Proof. By Lemma C.1, $f(T_*) = 0$ and f is irreducible. Its reductions modulo 29, 7 and 131 are squarefree, with factor degrees (8), (7, 1) and (5, 1, 1, 1) respectively. Dedekind's theorem [6, Chapter 13] gives permutations of these cycle types in $\text{Gal}(f)$. Irreducibility makes this group transitive. The seven-cycle makes a point stabilizer transitive on the other seven roots. Thus $\text{Gal}(f)$ is doubly transitive and hence primitive. Jordan's theorem states that a primitive group of degree n containing a cycle of prime length $p \leq n - 3$ contains A_n [7, Theorem 3.3E], [20, Theorem 13.9]. Applying it with $p = 5$ gives $A_8 \subseteq \text{Gal}(f)$. The odd eight-cycle then gives $\text{Gal}(f) = S_8$. Lemma C.2 counts the real zeros.

In the splitting field, F is the fixed field of a point stabilizer S_7 . Any larger subgroup moves the eighth point and is transitive. Its order is at least $8 \cdot 7!$, so it is S_8 . Galois correspondence excludes proper intermediate fields. If T_* lay in a radical extension of $\mathbb{Q}$, all its conjugates would lie in the Galois closure of that extension. In characteristic zero this closure has solvable Galois group [6, Chapter 8], which would make $\text{Gal}(f) = S_8$ a quotient of a solvable group. This contradiction excludes a radical expression for T_* . $\square$

At the minimum, let A_o be the horizontal component of the first period vector in Appendix B, and let γ_o be the square density of that torus. Since $g_B = 0$, Proposition B.1 gives

$$A_o = 1 + x_{0*} + \frac{1 - c_*(T_* - 3)}{s_*}, \quad \gamma_o = \frac{11}{A_o(T_* - 1) + T_*},$$

where the starred quantities are the values at (u_*, T_*) .

Corollary 8.2 (The field of the minimizing parameters). *The minimizing parameters generate the same field,*

$$F = \mathbb{Q}(u_*) = \mathbb{Q}(c_*) = \mathbb{Q}(w_*) = \mathbb{Q}(A_o) = \mathbb{Q}(\gamma_o),$$

and $\sqrt{5} \notin F$, so $[F(\sqrt{5}) : F] = 2$.

Proof. The formula $T_* = (6u_* + 4)/(1 + 2u_* - u_*^2)$ gives $F \subseteq \mathbb{Q}(u_*)$. Since $P(u_*) = 0$ and $\deg P = 8 = [F : \mathbb{Q}]$, the fields are equal. Lemma C.3 expresses c_*, w_*, A_o, γ_o as nonconstant rational functions of u_* with numerator and denominator degrees less than eight. A rational value would, after clearing the nonzero denominator, give a nonzero polynomial of degree less than eight vanishing at u_* . Each value therefore generates F , which has no proper intermediate fields. This also excludes $\mathbb{Q}(\sqrt{5})$. $\square$

Corollary 8.3 (Adjoining $\sqrt{5}$). *Let K be the splitting field of f . Then*

$$K \cap \mathbb{Q}(\sqrt{5}) = \mathbb{Q}, \quad \text{Gal}(K(\sqrt{5})/\mathbb{Q}) \cong S_8 \times C_2,$$

where C_2 is the cyclic group of order two. In particular, F and $\mathbb{Q}(\sqrt{5})$ are linearly disjoint over $\mathbb{Q}$. The eigenvalues $1 + w_\varphi$ and $1 + w_*\psi$ of M_{w_*} lie in $F(\sqrt{5}) \setminus F$.*

Proof. Every index-two subgroup of S_8 is the kernel of a surjective homomorphism $S_8 \rightarrow C_2$. The transpositions are conjugate and generate S_8 , so they all have the same nontrivial image. The homomorphism is therefore the sign map. Thus A_8 is the unique index-two subgroup and $E = K^{A_8}$ the unique quadratic subfield of K .

Complex conjugation fixes the two real roots and exchanges the other six in three pairs. Its odd permutation acts nontrivially on E , so E is imaginary and cannot equal $\mathbb{Q}(\sqrt{5})$. Hence $K \cap \mathbb{Q}(\sqrt{5}) = \mathbb{Q}$ and $[K(\sqrt{5}) : K] = 2$. The compositum is Galois, and restriction embeds its

Galois group into $\text{Gal}(K/\mathbb{Q}) \times \text{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. Both groups have order $2 \cdot 8!$, so restriction is an isomorphism. Linear disjointness follows, also for $F \subset K$.

Since $w_* \in F \setminus \{0\}$, either eigenvalue belonging to F would imply $\varphi \in F$ or $\psi \in F$, contrary to $\sqrt{5} \notin F$. $\square$

Corollary 8.4 (The first nonradical packing side). *If $s(11) = T_*$, then 11 is the least positive integer n for which $s(n)$ is not expressible in radicals over $\mathbb{Q}$.*

Proof. For $1 \leq n \leq 10$, the known minimum sides are integers except for $s(5) = 2 + 1/\sqrt{2}$ and $s(10) = 3 + 1/\sqrt{2}$ [9, 15, 18]. All these numbers are expressible in radicals, whereas T_* is not by Theorem 8.1. $\square$

APPENDIX A. RATIONAL BOUNDS REDUCE FEASIBILITY TO TWO CLEARANCES

Throughout this appendix (u, T) ranges over $\mathcal{R} = [9/25, 37/100] \times [387/100, 777/200]$, and every terminating decimal denotes an exact rational number. Define

$$\begin{aligned} m_{\pm} &= c \pm s, & p_1 &= 1 - s + s\rho, & p_2 &= 1 - c\rho, & p_3 &= 1 + c\eta + s\rho, \\ p_4 &= c - sm_-, & p_5 &= s + cm_-, & p_6 &= c(1 - \zeta) + 2s. \end{aligned} \quad (\text{A.1})$$

Lemma A.1 (Rational bounds). *On $\mathcal{R}$ each quantity in Table 5 lies strictly between its two displayed bounds, which are given in thousandths.*

Bound	c	s	m_+	m_-	ρ	η	x_0	ζ	p_1	p_2	p_3	p_4	p_5	p_6
Lower	759	637	1407	108	318	10	2018	499	556	737	1210	671	718	1584
Upper	771	651	1411	134	340	40	2050	591	580	759	1253	703	755	1689

TABLE 5. Strict lower and upper bounds, in thousandths, for the parameters in (2.1), (2.2) and (A.1), valid throughout $\mathcal{R}$.

Proof. The functions $c = (1 - u^2)/(1 + u^2)$ and $m_- = c - s$ decrease on $\mathcal{I}$, while $s = 2u/(1 + u^2)$ and $m_+ = c + s$ increase. For the last assertion, $m'_+ = 2(1 - 2u - u^2)/(1 + u^2)^2 > 0$. The endpoint values give

$$\frac{8631}{11369} \leq c \leq \frac{272}{353}, \quad \frac{225}{353} \leq s \leq \frac{7400}{11369}, \quad \frac{497}{353} \leq m_+ \leq \frac{16031}{11369}, \quad \frac{1231}{11369} \leq m_- \leq \frac{47}{353}.$$

Since $\rho = 1 - c(T - 3)$ with $.870 \leq T - 3 \leq .885$, its extreme values occur at opposite corners of $\mathcal{R}$. They give $2807/8825 \leq \rho \leq 386003/1136900$. The numerator $n_\eta = 2c - 1 - c^2t$ of η satisfies $\partial_c n_\eta = 2(1 - ct) > 0$ and $\partial_t n_\eta = -c^2 < 0$. The bounds on c and t therefore give $.008 < n_\eta < .025$ and $\eta = n_\eta/s \in (.010, .040)$. Differentiation gives

$$\partial_u x_0 = -\frac{2\{(T - 2)(1 + u^2) - 4u\}}{(1 - u^2)^2} < 0, \quad \partial_T x_0 = -\frac{2u}{1 - u^2} < 0,$$

where the expression in braces exceeds $1.87 - 1.48 > 0$. Also,

$$\partial_u \zeta = -\frac{T(u^4 - u^3 + u + 1) - 2u^3 - 4u - 4}{2u^3} > 0, \quad \partial_T \zeta = \frac{(1 + u^2)(1 + 2u - u^2)}{4u^2} > 0,$$

where $u^4 - u^3 + u + 1 > 0$ and $T < 4$ give $T(u^4 - u^3 + u + 1) - 2u^3 - 4u - 4 < 2u^3(2u - 3) < 0$. The corner values give

$$\frac{17420}{8631} \leq x_0 \leq \frac{2229}{1088}, \quad \frac{2530667}{5062500} \leq \zeta \leq \frac{6465493103}{10952000000},$$

which proves the first eight bounds in Table 5. Substitution in (A.1) bounds $p_1, \dots, p_6$ by interval arithmetic. For example,

$$1 + .759(.010) + .637(.318) < p_3 < 1 + .771(.040) + .651(.340).$$

$\square$

Lemma A.2 (Containment). *All eleven squares lie in $[0, T]^2$. Exactly eleven of their forty-four wall clearances vanish identically, and each of the other thirty-three exceeds $1/2$ on $\mathcal{R}$.*

Proof. The axis-parallel bounding box of $G(U(A, B))$ has lower corner

$$(1 + cA - s(B + 1 - \rho), 1 + sA + c(B - \rho))$$

and both side lengths equal to m_+ . With $s\eta = c(1 + \rho) - 1$ for the bottom of K_7 and the definition of ζ for the right side of K_{10} , this gives the eleven bounding boxes in Table 6.

i	0	1	2	3	4	5	6	7	8	9	10
$x_{\min}$	0	$T - 1$	x_0	0	1	0	p_1	p_3	$p_1 + p_4$	$p_3 + p_4$	$T - m_+$
$y_{\min}$	0	0	$T - 1$	$T - 1$	$T - 1$	$T - 2$	p_2	0	$p_2 + p_5$	p_5	p_6
d	1	1	1	1	1	1	m_+	m_+	m_+	m_+	m_+

TABLE 6. Axis-parallel bounding boxes of the eleven squares K_i . Each box is $[x_{\min}, x_{\min} + d] \times [y_{\min}, y_{\min} + d]$, with parameters as in (2.2) and (A.1).

A bounding box $(x, y) + [0, d]^2$ has wall clearances $x, y, T - x - d$ and $T - y - d$. For the six axis-parallel squares, exactly nine vanish and the rest exceed $1/2$. Substitution of Table 5 gives Table 7 for the five rotated squares. The smallest nonzero bound is $3.870 - 1.253 - .703 - 1.411 = .503$.

i	Left	Bottom	Right	Top
6	.556	.737	1.879	1.700
7	1.210	0	1.206	2.459
8	1.227	1.455	1.176	.945
9	1.881	.718	.503	1.704
10	2.459	1.584	0	.770

TABLE 7. Wall clearances of the rotated squares K_i in $[0, T]^2$. Nonzero clearances strictly exceed the listed bounds throughout $\mathcal{R}$; zeros denote identically vanishing clearances.

□

A.1. Separating axes account for all fifty-five pairs. Let $e_x = (1, 0)$ and $e_y = (0, 1)$, and define

$$\mathcal{N} = \{\pm e_x, \pm e_y, \pm e_1, \pm e_2\}, \quad \Delta_{ij}(n) = \ell_j(n) - r_i(n).$$

By Lemma 2.1 the interiors of K_i and K_j are disjoint exactly when $\Delta_{ij}(n) \geq 0$ for at least one $n \in \mathcal{N}$.

Lemma A.3 (The fifty-three inactive pairs). *The pairs and normals of Table 9 satisfy the stated positive bounds or identities throughout $\mathcal{R}$.*

Proof. For an axis-parallel square $U(x, y)$, the lower e_1 and e_2 projections are $cx + sy$ and $-s(x + 1) + cy$. Each projection interval has length m_+ . For $G(U(A, B))$, the lower projections are $m_+ + A$ and $m_- + B - \rho$, and each interval has length one. The e_x and e_y projections follow from Table 6. Subtracting endpoints gives the positive-clearance formulas in Table 8.

The stated bounds follow from Table 5 and $.870 \leq t \leq .885$. The zero-clearance identities in Table 9 follow from the same projection formulas. □

n	Possible values of $\Delta_{ij}(n)$
e_x	$1 + t, p_3 - 1, T - 1 - m_+, p_1 + p_4 - 1, p_3 + p_4 - 1$
$-e_x$	$T - 1 - p_1 - m_+, T - 1 - p_3 - m_+, x_0 - 1, x_0 - 2$
e_y	$1 + t, t, p_6 - 1$
$-e_y$	$T - 1 - m_+, T - 1 - p_5 - m_+, T - 1 - p_2 - m_+$
e_1	$1, 2, \eta, 1 + \eta, 2 + \eta$
$-e_1$	1 (the constant clearance of the pair $(2, 6)$)
e_2	$m_+(1 + t), m_+(1 + t) - s, m_+(1 + t) - c, m_+t + c - 1, m_-$
$-e_2$	$c, 1 + c, s, 1 + s, 1, m_- + \zeta$

TABLE 8. Possible positive clearances by normal for the inactive pairs in Table 9. Here $\Delta_{ij}(n) = \ell_j(n) - r_i(n)$, $t = T - 3$ and $m_{\pm} = c \pm s$.

n	Pairs with $\Delta_{ij}(n) > 0$	Pairs with $\Delta_{ij}(n) = 0$	Lower bound
e_x	$(0, 1), (0, 7), (3, 10), (5, 8), (5, 9), (5, 10)$	$(3, 4)$	.210
$-e_x$	$(1, 6), (1, 7), (2, 3), (2, 4), (2, 5)$	$(4, 5)$	.018
e_y	$(0, 3), (0, 4), (0, 5), (1, 2), (1, 10)$	—	.584
$-e_y$	$(2, 7), (2, 9), (4, 6), (4, 7)$	$(3, 5)$	.700
e_1	$(0, 2), (0, 8), (0, 9), (0, 10),$ $(6, 9), (6, 10), (7, 10), (8, 10)$	$(0, 6), (6, 8), (7, 9), (9, 10)$	.010
$-e_1$	$(2, 6)$	$(2, 8)$	1
e_2	$(1, 3), (1, 4), (1, 5), (1, 8), (7, 8)$	—	.108
$-e_2$	$(3, 6), (3, 7), (3, 8), (3, 9), (4, 9), (4, 10), (5, 7)$	$(4, 8), (5, 6), (6, 7), (8, 9)$	.607

TABLE 9. The fifty-three inactive pair conditions consist of forty-one positive clearances and twelve identities throughout $\mathcal{R}$. Each positive clearance $\Delta_{ij}(n) = \ell_j(n) - r_i(n)$ is at least the stated lower bound. The pair (i, j) has $i < j$, and n points from K_i toward K_j .

Lemma A.4 (The two active pairs). *Among the normals in $\mathcal{N}$, only e_2 can separate the pair $(1, 9)$ and only $-e_2$ can separate the pair $(2, 10)$. The retained clearances are*

$$g_A = m_+t - (2 - c), \quad g_B = c + \zeta - sx_0,$$

and for each pair the other seven clearances with normals in $\mathcal{N}$ are strictly negative throughout $\mathcal{R}$.

Proof. The projection formulas give the expressions in Table 10. For the first six normals, substitution of Table 5 gives the displayed strict lower bounds.

The bounds in Table 5 also give $|g_A| < .020$ and $|g_B| < .080$. Indeed, $g_A > 1.407(.870) + .759 - 2 = -.01691$ and $g_B > .759 + .499 - .651(2.050) = -.07655$. Reversing the endpoints gives the upper bounds. The two remaining margins therefore exceed 2.327. This proves all fourteen signs. $\square$

A.2. Small window changes preserve the family minimum.

Lemma A.5 (Independence of the endpoints). *There are open intervals $\mathcal{I}^+ \supset \mathcal{I}$ and $\mathcal{J}^+ \supset \mathcal{J}$ such that the squares of Table 1 form a packing in $[0, T]^2$ exactly when $T \geq \max\{T_A(u), T_B(u)\}$, for $(u, T) \in \mathcal{I}^+ \times \mathcal{J}^+$. For every closed rectangle in $\mathcal{I}^+ \times \mathcal{J}^+$ that contains (u_*, T_*) , the feasible family has the same unique minimum (u_*, T_*) .*

Proof. The eleven zero wall clearances and the twelve zero pair clearances are algebraic identities. By the preceding lemmas, the other thirty-three wall clearances, the forty-one positive pair clearances and the negatives of the fourteen excluded clearances have positive minima on the compact rectangle $\mathcal{R}$. They retain their signs on a product neighborhood of $\mathcal{R}$. Shrink this

n	$-\Delta_{1,9}(n)$	Lower bound	$-\Delta_{2,10}(n)$	Lower bound
	Expression		Expression	
e_x	$T - p_3 - p_4$	1.914	$x_0 + 1 + m_+ - T$	.540
$-e_x$	$p_3 + p_4 + m_+ - T + 1$	.403	$T - x_0$	1.820
e_y	$1 - p_5$	.245	$T - p_6$	2.181
$-e_y$	$p_5 + m_+$	2.125	$p_6 + m_+ - T + 1$	.106
e_1	$c(T - 1) - \eta - 1$	1.138	$m_+ - \eta$	1.367
$-e_1$	$m_+ + \eta + 2 - c(T - 1)$	1.192	$1 + \eta$	1.010
e_2	—	—	$m_+ + 1 + g_B$	2.327
$-e_2$	$m_+ + 1 + g_A$	2.327	—	—

TABLE 10. Exact expressions and strict lower bounds on $\mathcal{R}$ for the negatives of the alternative clearances, with $\Delta_{ij}(n) = \ell_j(n) - r_i(n)$. Dashes denote the retained directions e_2 for $(1, 9)$ and $-e_2$ for $(2, 10)$.

neighborhood so that all denominators remain nonzero and the strict inequalities $c > s > 0$, $D > 0$ and $T'_A > 0 > T'_B$ persist. The eight-normal separation argument of Lemma 2.2 and the identities (2.5) give the feasibility criterion there. The two bounds still cross only at u_* . For $u < u_*$ we have $T_B(u) > T_*$, and for $u > u_*$ we have $T_A(u) > T_*$. At $u = u_*$, feasibility gives $T \geq T_*$. Thus every other feasible point has a larger enclosing side. $\square$

A.3. Rational intervals isolate the minimizing parameters.

Proposition A.6 (Isolating intervals). *Let*

$$\underline{u} = \frac{3657693076046772933885}{10^{22}}, \quad \bar{u} = \frac{3657693076046772933886}{10^{22}}.$$

Then $\underline{u} < u_ < \bar{u}$ and $3.877083590022814177307 < T_* < 3.877083590022814177308$.*

Proof. Substitution into P gives $-3 < 10^{22}P(\underline{u}) < -2$ and $2 < 10^{22}P(\bar{u}) < 3$. Hence the unique zero of P in $\mathcal{I}$ lies between $\underline{u}$ and $\bar{u}$. Since T_A is strictly increasing, the rational values $T_A(\underline{u})$ and $T_A(\bar{u})$ bracket T_* . Both lie in the stated interval. $\square$

APPENDIX B. THE ELEVEN-SQUARE MOTIF ADMITS A DENSER PERIODIC EXTENSION

For $(u, T) \in \mathcal{P}$ define

$$A_o = 1 + \frac{c + \zeta + \rho}{s}, \quad a_o = (A_o, 1), \quad b_o = (A_o - T, T), \quad \Lambda_o = \mathbb{Z}a_o + \mathbb{Z}b_o. \quad (\text{B.1})$$

The value of A_o makes the supporting lines of K_{10} and $K_0 + a_o$ in direction $-e_2$ coincide. The other two period conditions are $(a_o)_y = 1$ and $a_o - b_o = (T, 1 - T)$. The covolume of Λ_o is

$$\det(a_o, b_o) = A_o(T - 1) + T. \quad (\text{B.2})$$

Proposition B.1 (Periodic continuation). *For every $(u, T) \in \mathcal{P}$, the translates $K_i + \nu$, $0 \leq i \leq 10$, $\nu \in \Lambda_o$, have pairwise disjoint interiors. The eleven squares embed in the quotient $\mathbb{R}^2/\Lambda_o$, and*

$$A_o = 1 + x_0 + \frac{\rho}{s} + \frac{g_B}{s},$$

so that $A_o = 1 + x_0 + \rho/s$ when $g_B = 0$.

Proof. By Table 5,

$$1 + \frac{759 + 499 + 318}{651} < A_o < 1 + \frac{771 + 591 + 340}{637}.$$

Hence

$$\frac{17}{5} < A_o < \frac{37}{10} < T < 4, \quad 0 < d_o < \frac{1}{2}, \quad d_o = T - A_o. \quad (\text{B.3})$$

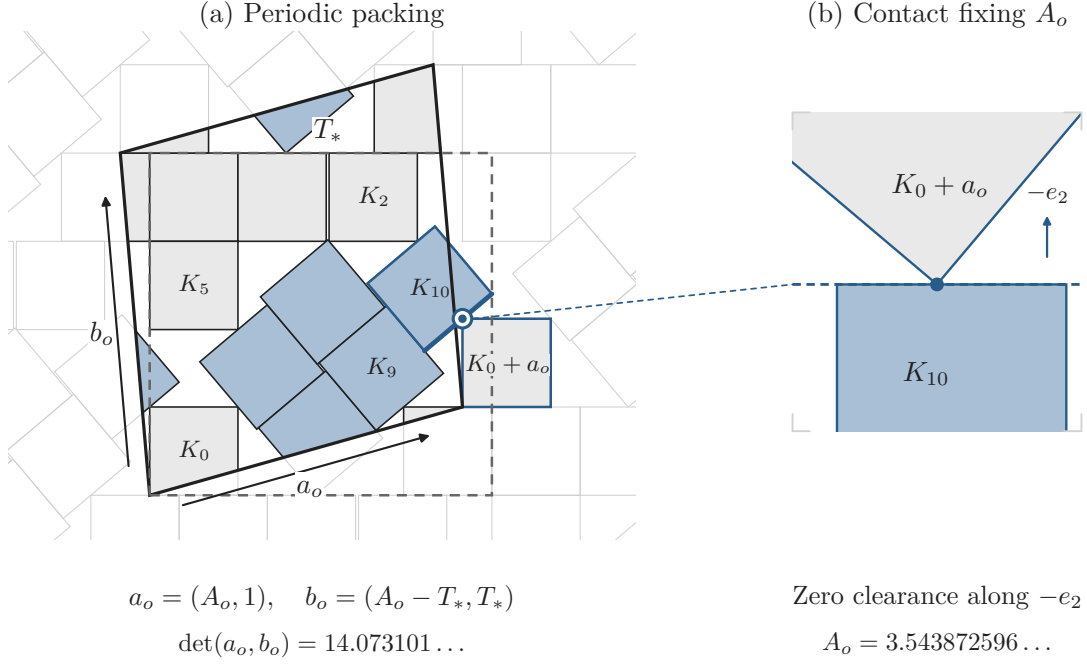


FIGURE 6. The periodic continuation of Trump's configuration. The solid parallelogram bounds one period, and the enclosing square $[0, T_*]^2$ is dashed. Neighboring copies have light gray outlines. Tilted squares are blue and axis-parallel squares are light gray. The contacting copies K_{10} and $K_0 + a_o$ have dark blue outlines, with their contact enlarged at right in the orthonormal frame $(-e_1, -e_2)$. The common support line and contact point are dark blue. Zero clearance in direction $-e_2$ determines A_o .

The Gram matrix of a_o and b_o minus $9I$ has diagonal entries $A_o^2 - 8 > 89/25$ and $T^2 + d_o^2 - 9 > 59/10$, since $T \geq 387/100$. Its off-diagonal entry satisfies $0 < T - A_o d_o < 4$, so its determinant exceeds $(89/25)(59/10) - 16 = 1251/250 > 0$. The matrix is positive definite, giving

$$|ia_o + jb_o| > 3\sqrt{i^2 + j^2} \quad ((i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}). \quad (\text{B.4})$$

If $(i, j) \notin \{-1, 0, 1\}^2$, the translation has length greater than $6 > \sqrt{2}T$, the diameter of $[0, T]^2$. The configuration and its translate cannot meet. Among the remaining translations, b_o , $a_o - b_o$ and $a_o + b_o$ separate the interiors across the lines $y = T$, $x = T$ and $y = T + 1$, respectively. Exchanging the two copies handles the negative translations. It remains to consider a_o .

For the translation by a_o , use the bounding boxes in Table 6. The bounds in Table 5 give $.556 < p_1 < 1$, $p_3 > 1$, $0 < p_4 < c < .771$, $m_+ < 1.411$ and $2 < x_0 < 2.1$. The gap identity (2.5) gives

$$p_3 = T - 2 - s - \frac{gA}{s} \leq T - 2 - s,$$

so $p_3 + p_4 + m_+ \leq T - 2 + 2c - s(c - s) < 1.885 + 2(.771) - .637(.108) < 3.4$. The other right endpoints, except that of K_{10} , are also less than 3.4: we have $p_1 + p_4 + m_+ < 3.4$ since $p_1 < 1$, and $p_3 + m_+ \leq T - 2 + c < 2.656$. Including the axis-parallel squares gives

$$\max_{i \notin \{1, 10\}} r_i(e_x) < 3.4, \quad \min_{j \notin \{0, 3, 5\}} \ell_j(e_x) > .5, \quad (\text{B.5})$$

where the second inequality uses $p_1 > .556$, $p_3 > 1$, $p_4 > 0$ and $T - m_+ > 2$. Since $A_o > 3.4$ and $A_o + .5 > 3.9 > T$, only pairs with $i \in \{1, 10\}$ and $j \in \{0, 3, 5\}$ can meet. The line $y = 1$ separates the interior of K_1 from that of $K_0 + a_o$. The squares $K_3 + a_o$ and $K_5 + a_o$ lie on or above $y = T - 1 > 1$, so they also have interiors disjoint from K_1 . The line $y = T$ separates K_{10} from $K_3 + a_o$. The clearance of K_{10} against $K_0 + a_o$ in direction $-e_2$ is

$$sA_o - 2c - (s - c + \zeta + \rho) = 0,$$

and the clearance of K_{10} against $K_5 + a_o$ in the direction e_1 is

$$cA_o + s(T - 1) - (c + s + \eta + 3) = \frac{(1 - c)(1 - s)}{s} + \frac{c}{s}g_B > 0, \quad (\text{B.6})$$

since $0 < c, s < 1$ and $g_B \geq 0$. All six pairs have disjoint interiors, proving periodic nonoverlap. Each unit square embeds in the quotient because its diameter $\sqrt{2}$ is less than every nonzero period length, by (B.4). The identity $g_B = c + \zeta - sx_0$ gives the formula for A_o . $\square$

At the minimum both clearances vanish and $A_o = 1 + x_{0*} + \rho_*/s_*$. The period area and the density of the eleven squares in the period torus are

$$A_o(T_* - 1) + T_* = 14.07310128\dots, \quad \gamma_o = \frac{11}{A_o(T_* - 1) + T_*} = 0.78163297\dots$$

The enclosing square has area $T_*^2 \approx 15.03178$ and square density $11/T_*^2 = 0.73178\dots$. These values follow from the rational functions in Lemma C.3 and the interval in Proposition A.6.

B.1. The marked period torus recovers the packing. A *marked period torus* is a Euclidean flat torus with an ordered period basis (a_o, b_o) , an oriented orthonormal frame, and eleven labelled embedded unit squares with disjoint interiors. It also carries coherent lifts of the contact diagram below and of the four reference lines $x = 0, x = T, y = 0, y = T$. Here $T = b_o \cdot e_y$, and the lift of the lower left corner of K_0 is the origin.

- (1) The six axis-parallel squares have lower left corners $(0, 0), (T - 1, 0), (x_0, T - 1), (0, T - 1), (1, T - 1)$ and $(0, T - 2)$, with x_0 free.
- (2) The five tilted squares share a frame (e_1, e_2) and have, relative to the lower left corner of K_6 , lower left coordinates $(0, 0), (\eta, -1), (1, v), (\eta + 1, v - 1)$ and $(\eta + 2, -\zeta)$ in that frame, with η, v and ζ free.
- (3) Their supporting lines satisfy $\ell_6(e_1) = r_0(e_1)$, $\ell_6(-e_2) = r_5(-e_2)$, $\ell_8(-e_2) = r_4(-e_2)$ and $\ell_8(-e_1) = r_2(-e_1)$. The wall incidences are $\ell_7(e_y) = 0$ and $r_{10}(e_x) = T$.
- (4) The periods satisfy $(a_o)_y = 1$ and $(b_o)_y = (a_o)_x - (b_o)_x = T$, and the supporting lines of K_{10} and $K_0 + a_o$ in the direction $-e_2$ coincide.
- (5) The measured pair (u, T) lies in $\mathcal{R}$.

The marking includes the chosen lifts, specified by finitely many paths from the marked origin up to homotopy relative to both endpoints. The reference square need not embed in the torus. Let $\mathcal{T}^{\text{per}}$ be the set of marked period tori up to frame-preserving isometries that preserve all marks.

Proposition B.2 (Recovery of the configuration). *Quotienting a member of $\mathcal{P}$ by Λ_o defines a bijection from $\mathcal{P}$ onto $\mathcal{T}^{\text{per}}$. Its inverse lifts the marked contact diagram, and both maps preserve $T = b_o \cdot e_y = (a_o - b_o) \cdot e_x$.*

Proof. Proposition B.1 gives the forward map. Conversely, the periods determine T and the common tilted frame determines u . The first supporting-line equality places the lower left corner of K_6 at $(1, 1) - \rho e_2$ for a unique real ρ . The second gives $\rho = 1 - c(T - 3)$. The remaining incidences give, in order,

$$v = c - s, \quad s\eta = (1 + \rho)c - 1, \quad cx_0 + s(T - 3) = c + 2 - s, \quad s(\zeta + \rho) + (3 + \eta)c = T - 1.$$

Since $c, s > 0$, these equations determine all coordinates and reproduce Table 1. The period contact gives $(a_o)_x = A_o$, and the other period conditions determine a_o and b_o . An interior intersection of two lifted squares would project to one in the torus, so the lifts have disjoint interiors. By Lemma 2.2, they lie in $[0, T]^2$. The reconstructed pair therefore belongs to $\mathcal{P}$. The constructions preserve every mark and are inverse. The period conditions give both expressions for T . $\square$

APPENDIX C. EXACT ARITHMETIC DETERMINES THE SIDE POLYNOMIAL AND ITS ROOTS

Lemma C.1 (An octic relation and three factorizations). *Let P be the polynomial (2.7) and f the polynomial (8.1). Then $f(T_*) = 0$ and f is irreducible over $\mathbb{Q}$. Its reductions modulo 29, 7 and 131 are squarefree, with irreducible factors of degrees (8), (7, 1) and (5, 1, 1, 1) respectively.*

Proof. Define

$$\mathcal{C}(u) = -1373u^8 + 4318u^7 + 15386u^6 + 29302u^5 + 16044u^4 + 4914u^3 + 1414u^2 + 186u - 31. \quad (\text{C.1})$$

Expansion gives

$$(1 + 2u - u^2)^8 f\left(\frac{6u + 4}{1 + 2u - u^2}\right) = P(u) \mathcal{C}(u).$$

Since $P(u_*) = 0$ and $1 + 2u_* - u_*^2 > 0$, this gives $f(T_*) = 0$.

In the indicated prime fields define

$$\begin{aligned} f_7(X) &= X^7 + 2X^6 - 2X^5 + 3X^4 + X^3 + 2X^2 + 3X - 2, \\ f_{29}(X) &= X^8 + 9X^7 + 4X^6 - X^5 + 9X^4 - 3X^3 + 3X^2 + 8X + 8, \\ f_{131}(X) &= X^5 - 50X^4 - 13X^3 - 37X^2 - 58X + 50. \end{aligned}$$

Multiplication gives

$$\begin{aligned} f &\equiv (X - 1)f_7 \pmod{7}, & f &\equiv f_{29} \pmod{29}, \\ f &\equiv (X + 30)(X + 55)(X - 55)f_{131} \pmod{131}. \end{aligned} \quad (\text{C.2})$$

For irreducibility, define

$$\begin{aligned} h_7(X) &= -X^5 + X^4 + 3X^3 + 3X^2 + 3X + 3, \\ h_{29}(X) &= -8X^7 + 8X^6 + 6X^5 - 11X^4 + 6X^3 - 10X^2 - 6X - 9, \\ h_{131}(X) &= -64X^4 + 57X^3 - 62X^2 + 55X + 17. \end{aligned}$$

Repeated reduction in $\mathbb{F}_p[X]/(f_p)$ gives the congruences

$$X^{p^{n_p}} \equiv X, \quad h_p(X)(X^{p^{j_p}} - X) \equiv 1 \pmod{f_p}, \quad (\text{C.3})$$

where (n_p, j_p) is (7, 1), (8, 4) and (5, 1) for $p = 7, 29$ and 131.

An irreducible polynomial of degree d over $\mathbb{F}_p$ divides $X^{p^n} - X$ exactly when $d \mid n$. Thus (C.3) forces every irreducible factor of f_p to have degree dividing n_p but not j_p . For $n_p = 5, 7$ this excludes the only proper divisor 1. For $n_p = 8$ it excludes 1, 2, 4. Each f_p is therefore irreducible. The distinct linear factors in (C.2) cannot divide the higher-degree irreducible factors, so the reductions are squarefree. Irreducibility modulo 29 proves that the monic polynomial f is irreducible over $\mathbb{Q}$. $\square$

Lemma C.2 (The two real zeros). *The polynomial f has exactly one negative and one positive real zero.*

Proof. Define the rational Sturm sequence by $S_0 = f$, $S_1 = f'$ and $S_{i+1} = -\text{rem}(S_{i-1}, S_i)$. Exact division gives the degrees and signs in Table 11. The sequence ends in a nonzero constant and no entry vanishes at zero, so Sturm's theorem gives one zero in $(-\infty, 0)$ and one in $(0, \infty)$. $\square$

i	0	1	2	3	4	5	6	7	8	Sign changes
$\deg S_i$	8	7	6	5	4	3	2	1	0	
$x = -\infty$	+	-	-	+	-	+	+	-	-	5
$x = 0$	-	+	+	+	-	+	+	-	-	4
$x = +\infty$	+	+	-	-	-	-	+	+	-	3

TABLE 11. Degrees and signs of the rational Sturm sequence of f .

Lemma C.3 (Rational formulas for the minimizing parameters). *The following nonconstant rational functions have numerator and denominator of degree at most six. Their values at u_* are the parameters c_*, w_*, A_o, γ_o of Corollary 8.2, respectively:*

$$c(u) = \frac{1 - u^2}{1 + u^2}, \quad w(u) = \frac{2u(1 - u^2)}{(1 + u^2)^2}, \quad \mathcal{A}(u) = -\frac{u^5 + u^4 + 2u^3 + 8u^2 - 3u - 5}{(u - 1)(u + 1)(u^2 - 2u - 1)},$$

$$\Gamma(u) = \frac{11(u - 1)(u^2 - 2u - 1)^2}{(u + 1)(u^5 + 3u^4 - 4u^3 + 32u^2 - 5u - 19)}.$$

Proof. The first two formulas are the half-angle identities. Since $g_B = 0$ at the minimum, Proposition B.1 gives

$$A_o = 1 + \frac{c_* + 2 - s_* - s_*(T_* - 3)}{c_*} + \frac{1 - c_*(T_* - 3)}{s_*}.$$

Substituting the half-angle formulas and $T_* = (6u_* + 4)/(1 + 2u_* - u_*^2)$ gives $A_o = \mathcal{A}(u_*)$, and then $\gamma_o = \Gamma(u_*)$. The displayed polynomials have degree at most six, with no proportional numerator–denominator pair. The first three denominators do not vanish on $\mathcal{I}$. The last is a nonzero multiple of the positive period area at u_* . $\square$

APPENDIX D. ERDŐS'S SQUARE-PACKING CONJECTURE (PROBLEM 106)

Let $f(n)$ be the supremum of the sum of the side lengths of n squares contained in $[0, 1]^2$, with positive side lengths and pairwise disjoint interiors. Boundary contacts are permitted, and orientations are unrestricted. Erdős's square-packing conjecture [8], catalogued as Problem 106 [5], asks whether $f(k^2 + 1) = k$ for every positive integer k . The axis-parallel case was proved by Baek, Koizumi and Ueoro [3]. The following proof uses Sprite143's published seventeen-square layout [17], with a rational rotation.

Theorem D.1. *There are seventeen squares in the unit square with pairwise disjoint interiors and total side length*

$$4 + \frac{101}{1153304} > 4.$$

In particular, $f(17) > 4$.

Define the rational parameters

$$c = \frac{143}{145}, \quad s = \frac{24}{145}, \quad (a, h, b, r) = \frac{(1016863, 449731, 567132, 390485)}{2306608}. \quad (\text{D.1})$$

They satisfy $c^2 + s^2 = 1$, $b = a - h$ and $r(c + s) = h$. Let

$$d = \frac{1 - a}{2}, \quad x_0 = 1 - 3h, \quad y_0 = 1 - a, \quad \mathcal{U}(x, y; \ell) = [x, x + \ell] \times [y, y + \ell].$$

Table 12 specifies sixteen axis-parallel squares. The seventeenth is

$$R = (x_0 + rs, y_0) + \{v(c, s) + w(-s, c) : 0 \leq v, w \leq r\}. \quad (\text{D.2})$$

Since (c, s) and $(-s, c)$ are orthonormal, R is a square of side r . Its axis-parallel bounding square is

$$[1 - 3h, 1 - 2h] \times [1 - a, 1 - b]. \quad (\text{D.3})$$

Proof of Theorem D.1. The parameters in (D.1) satisfy

$$\frac{1}{4} < a < \frac{1}{2}, \quad 0 < h < b < \frac{3}{8}, \quad 1 - 3h > \frac{1}{4},$$

$$\frac{3}{4} < a + 2b < 1, \quad a + 2h < 1. \quad (\text{D.4})$$

Every side length in Table 12 is therefore positive. The B column ends at height $3a - 3/4 < 3/4$, and the C block fills $[a, 1] \times [0, 1 - a]$. The square D ends at height one and at abscissa $1 - 2b$. The E and F rows end at abscissa one, with vertical ranges $[1 - b, 1]$ and $[1 - a, 1 - b]$, respectively. These descriptions, (D.4) and (D.3) prove containment of all seventeen squares.

Label	Indices	Square	Count
L_j	$0 \leq j \leq 3$	$\mathcal{U}(0, j/4; 1/4)$	4
B_j	$0 \leq j \leq 2$	$\mathcal{U}(1/4, j(a - 1/4); a - 1/4)$	3
C_{ij}	$i, j \in \{0, 1\}$	$\mathcal{U}(a + id, jd; d)$	4
D	—	$\mathcal{U}(1/4, 1/4 + 2b; 3/4 - 2b)$	1
E_i	$i \in \{0, 1\}$	$\mathcal{U}(1 - 2b + ib, 1 - b; b)$	2
F_i	$i \in \{0, 1\}$	$\mathcal{U}(1 - 2h + ih, 1 - a; h)$	2

TABLE 12. The sixteen aligned squares; all indices are integers. Together with R in (D.2), these form the packing in Figure 7.

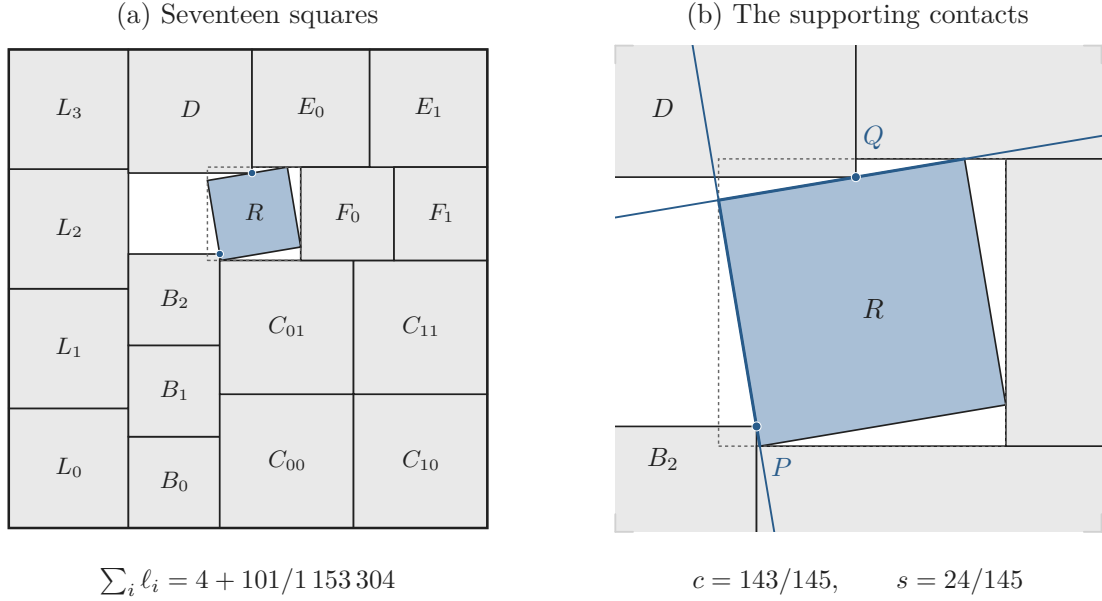


FIGURE 7. The seventeen-square packing and its two supporting contacts. The aligned squares are light gray and R is blue. The right panel enlarges P , the upper-right corner of B_2 , and Q , the lower-right corner of D . The dashed square is the bounding square (D.3); the supporting lines separate R from the entire B column and from D .

Squares within each aligned group have disjoint interiors by construction. The L column lies in $x \leq 1/4$, and all other squares lie in $x \geq 1/4$. Among the five remaining aligned groups, B is left of C, E, F , since $a \leq 1 - 2b$ and $a \leq 1 - 2h$, and below D , since $3a - 3/4 \leq 1/4 + 2b$. The block C is below D, E, F , since $1 - a \leq 1/4 + 2b$ and $1 - a < 1 - b$. Finally, D is left of E, F , since $b > h$, and F is below E . These are all ten pairings of those groups, and every inequality follows from (D.4) and $b = a - h$.

The bounding square (D.3) places R to the right of L , above C , below E and to the left of F . Only B and D remain. Their relevant corners are

$$P = (a, 3a - 3/4), \quad Q = (1 - 2b, 1/4 + 2b).$$

Direct substitution in (D.1) gives the two supporting-contact identities

$$\begin{aligned} ca + s(3a - 3/4) &= cx_0 + sy_0 + crs, \\ -s(1 - 2b) + c(1/4 + 2b) &= -sx_0 + cy_0 + rc^2. \end{aligned} \tag{D.5}$$

For $\alpha(x, y) = cx + sy$ and $\beta(x, y) = -sx + cy$, the positivity of c, s and (D.2) imply

$$\max_{B_0 \cup B_1 \cup B_2} \alpha = \alpha(P) = \min_R \alpha, \quad \min_D \beta = \beta(Q) = \max_R \beta.$$

The two lines in (D.5) therefore separate the interiors of R from the entire B column and from D . This completes the proof of pairwise non-overlap.

Summing the seventeen side lengths gives

$$\begin{aligned} S &= 1 + 3(a - 1/4) + 4d + (3/4 - 2b) + 2b + 2h + r \\ &= 3 + a + 2h + r = 4 + \frac{101}{1153304}. \end{aligned}$$

□

Corollary D.2. *For every integer $k \geq 4$,*

$$f(k^2 + 1) \geq k + \frac{101}{288326k} > k.$$

Proof. Divide the unit square into a $k \times k$ grid. Replace its lower-left 4×4 block by the packing of Theorem D.1, scaled by $4/k$, and retain the other $k^2 - 16$ grid squares. The interiors remain disjoint, the number of squares is $17 + k^2 - 16 = k^2 + 1$, and their total side length is

$$\frac{4}{k} \left(4 + \frac{101}{1153304} \right) + \frac{k^2 - 16}{k} = k + \frac{101}{288326k}.$$

This grid extension is also recorded in [17].

□

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